Mathematical Beliefs and Knowledge: A case study of a teacher's communication using representations

Andy Ash

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Institute of Education, University of Cumbria, UK

Cheshire and Wirral Maths Hub, UK

Abstract

Within education, the subject domain of mathematics is recontextualised, through a combination of teachers' practice and curriculum development, to become what I term 'school maths'. Currently, in English schools, there is a drive to reform school maths through introducing an approach that is known as 'teaching for mastery', which includes a strong emphasis on teachers using multiple representations. Despite this, it can be seen in the literature that *effective* use of representations in mathematics education is complex and requires a significant amount of teacher expertise. It is likely that teacher beliefs and knowledge will influence the way in which they use representations, however this is an area in need of further research. This study aims to contribute to better understanding the recontexualisation process by examining the relationship between teacher beliefs and knowledge, and their use of representations to teach fractions. Utilising a case study approach, a single teacher, Gillian, is the focus of this research. The aim being not to make generalised claims to knowledge, but instead contribute to theory development by adopting a critical realist methodology and using an innovative approach to analysis in applying Legitimation Code Theory (LCT). Data was collected from four interviews (two of which utilised a stimulated recall approach), two observed and video recorded lessons, and a textbook analysis. Findings show that studying beliefs and knowledge together, as a belief and knowledge system, is an effective way of understanding how teachers influence the recontextualisation process. Specifically, this study showed that believing in the importance of mathematical knowledge acquisition alongside the development of social learning, including learner dispositions, led to an effective use of representations, where dialogue was carefully used to negotiate the meaning of fractions. It was also found that LCT presents a useful way of explaining the organising principles of teacher beliefs, knowledge and practices and is an avenue for further research. Finally, the study found that the specific textbook used can play a key role in the recontextualization process.

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Acknowledgements

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Author's Declaration

I declare that this thesis is my own work and has not been submitted in substantially the same form for the award of higher degree elsewhere. The word-length conforms to the permitted maximum.

1 Introduction

In broad terms, this study is concerned with mathematical knowledge and its relation to primary school education, with teachers at the heart of this process. In this way, it is focussed upon the recontextualisation of knowledge through development of curriculum along with classroom practice, which can be seen as an intertwined process (Bernstein, 2000; Maton, 2014; Lilliedahl, 2015). Particularly, this study looks closely at an aspect of recontextualisation, where the subject domain of Mathematics becomes what might be termed ‘school maths’. Within this introduction, the background and rationale for this study will be set out, followed by introduction to the research context, aims and questions.

1.1 Background and Rationale

Integral to my study are three key areas; mathematics education research regarding use of representations, mathematical knowledge and beliefs, including how philosophical arguments about the nature of mathematics relate to classroom practice, and a social realist stance towards knowledge and the knowledge practices of teachers.

First, the use of representations in mathematics teaching features significantly within the research literature and is considered as an important part of teacher knowledge (Fennema and Franke, 1992). Specifically, there is broad consensus that using more than one type of representation (multiple representations) is an important part of successful mathematics teaching and learning (Goldin and Shteingold, 2001; Duval, 2006; Rau and Matthews, 2017). From an empirical perspective, both large scale meta-analysis and research reviews (Carbonneau, Marley and Selig, 2013; Henderson et al., ND) along with smaller scale studies (for example, Petersen and McNeil, 2013; Tunç-Pekkan, 2015) have shown that the use of different forms of representation can have a positive effect in relation to pupil learning of mathematical ideas. Taken individually, the smaller scale studies have often unearthed complexities involved in effective use of representations that are missed from larger studies and reviews. Such complexities are also identified within the theoretical literature where it is hypothesised that, whilst being able to use multiple representations is said to be a key tenant of deep mathematical understanding, using multiple representations to *teach* is tricky because mathematical objects lack ostensive referents: we can only communicate these ideas using representations of them (Duval, 2006). Therefore, the process by which representations are used, and meaning is negotiated in the classroom, is a complex and nuanced phenomenon that requires further study. My study focusses particularly on representation in the domain of fractions due to the significant body of research that can be built upon, along with the well documented difficulty that pupils and teachers often have with them (Cramer, Post and delMas, 2002; Charalambous and Pitta-Pantazi, 2007; Dreher and Kuntze, 2015; Gabriel et al., 2013; Hackenberg, 2013; Panaoura et al., 2009; Tunç-Pekkan, 2015).

Second, there is acknowledgement that teacher beliefs and knowledge are intertwined (Fennema and Franke, 1992; Kuntze, 2021) and one proposition of this study is that, together, they are likely to influence the way teachers use representations to negotiate mathematical meaning with pupils. This proposition draws partly from the literature on philosophical arguments about the nature of mathematics and partly from the empirical research. Within philosophical arguments about the nature of mathematics, there is a broad spectrum of beliefs where at one end there are ‘absolutist’ beliefs and at the other, ‘fallibilist’ beliefs (Ernest, 1991; Hersh, 1999). The former argument suggests that all mathematical knowledge exists externally to the human mind, whereas the latter suggests that it has been created by humans and is therefore subject to human error and is culturally and historically situated. These arguments about the nature of mathematics relate to representations as they call into question what is being represented (Radford, 2006). Is there such a thing as a singular mathematical object that exists externally to the human mind and can thus be represented ‘correctly’, or do the nature of mathematical objects vary from person to person and is mathematical meaning built through discourse using multiple representations? It is arguable that teachers' beliefs about the answers to these questions are likely to be an influencing factor on how representations are used.

Within the empirical research into teachers' beliefs and knowledge such questions have begun to be addressed, however studies have tended to focus upon either beliefs or knowledge rather than the two together (Shulman, 1986; Schoenfeld, 1989; Kloosterman and Cougan, 1994; Ma, 1999; De Corte, Op't Eynde and Verschaffel, 2004; Hill, Schilling and Ball, 2004; Kuntze, 2012; Sun, 2015). Studies focussed upon *beliefs* and their relationship to practice have yielded a variety of findings that often raise further questions for researchers due to the seemingly conflicting results. Namely, several studies have found there to be distinct variation between teachers' espoused beliefs and actual classroom practice (Erikson, 1993; Raymond 1997; Philipp, 2007). Equally, studies that have focussed upon teacher *knowledge* have found it to be an important factor in determining quality of instruction but also that beliefs are another important factor that can exert strong influence in this area, at times negating the influence of strong subject knowledge (Hill et al., 2008; Sleep and Eskelson, 2012; Beswick, 2012). However, despite this research, there is still very little known about how beliefs and knowledge influence teachers' use of representations, prompting questions about the processes involved in choosing and using representations. Therefore, another proposition of my study is that studying beliefs and knowledge together, rather than trying to separate them, may yield more useful findings.

Third, my study focusses upon the negotiation of mathematical meaning, therefore it is closely connected to the social realist movement that aims to develop understanding of knowledge and knowledge practices, and place this at the centre of sociological discussions about education (Moore and Young, 2009; Maton, 2014; Lilliedahl, 2015). This is, in part, a response to the significantly polarised state of education research, with positivist absolutism at one end and constructivist relativism at the other. Because of this polarisation, Maton (2014: 4) suggests that education research has either focussed on knowers or knowing rather than knowledge itself, leading to what he describes as “knowledge blindness”. Social realism attempts to avoid such a false dichotomy by placing knowledge at the heart of the debate, viewing it as socially constructed yet having an existence of its own, beyond subjectivism (Moore and Young, 2009; Maton, 2014). This is an important element of the theoretical framework of this study because understanding the knowledge practices of teachers will lead to a better understanding of what delineates success in the mathematics classroom. Aiding this aspect of the study is Maton’s (2014) Legitimation Code Theory (LCT) which draws upon the work of Bernstein and Bourdieu in attempting to create an explanatory framework that helps describe what organising principles underpin knowledge practices. In other words, by using and evaluating LCT as a tool to help study how teachers use representations in their maths lessons, I am attempting to avoid knowledge blindness and better understand what Bourdieu refers to as the “rules of the game” (Grenfell, 2014: 105).

1.2 Research Context

Within England, since the introduction of a new, ‘knowledge-rich’, National Curriculum in 2014 (DfE, 2013a; Gibb, 2021), there has been a drive to develop various curriculum reforms. This is part of a developing policy within the English education system where there is an unwavering emphasis on subject specific knowledge acquisition, and this is often associated with so called ‘teacher-led instruction’ (Gibb, 2017). This has been contrasted with ‘child-centred’ teaching and the two have been presented as dichotomous to one another (ibid., 2017). One of the most major curriculum reforms that has been on-going since 2014 in England is the introduction of ‘teaching for mastery’ in mathematics education. This is led by a national agency and is somewhat inspired by high-attaining East Asian regions such as Shanghai and Singapore (DfE, 2016). In essence, ‘teaching for mastery’ attests that all pupils are capable of learning school maths and that teaching should adopt approaches that enable this to happen (NCETM, 2023b). Because of this, many teachers across England are engaged in government funded professional development where the use of multiple representations in school maths is being promoted. This includes an emphasis on using manipulatives (physical materials designed for learning mathematical ideas) and the introduction of government recommended textbooks for primary schools (Boyd and Ash, 2018b). As a result of this, it is likely that many teachers across the country are broadening their use of representations in mathematics lessons, thus strengthening the need for more research in this area. Because of its importance to the contextual backdrop of my study, the idea of 'teaching for mastery' is outlined in further detail within the Literature Review chapter (section 2.4.1). In addition to this, another important contextual factor to my study was the international Coronavirus pandemic, which lead to nation-wide school closures. I conducted the data collection for my study before the pandemic struck but had initially intended on including more than one teacher, as a multiple case study. However, the advent of the pandemic caused me to re-think my study design and, ultimately, helped steer me in the direction of an approach that was more befitting for my research question. This is discussed further in the methodology chapter (section 4.3).

This study aligns with my current role as leader of a Maths Hub, which is the vehicle through which government funding for teaching for mastery is being allocated. In this role I have spent considerable time working alongside teachers, supporting the development of classroom practice, and gaining first-hand experience of these mathematics education reforms, through collaborative professional inquiry. This, alongside still regularly teaching myself, has provided me with valuable insight into some of the key issues arising within schools and, because of this, I consider myself to be an "insider" researcher (Hellawell, 2006: 484). This study also builds on my involvement in a collaborative action research project with teacher researchers and a university-based research mentor which focussed upon teachers using a new textbook scheme to develop their classroom practice (Boyd and Ash, 2018a, 2018b). During this research we found that there were important changes going on with regards to actual classroom practice alongside teacher beliefs about mathematics and mathematics teaching, and that these were closely tied to their experiences of professional development and using a textbook (ibid., 2018a, 2018b). Because of this influence exerted by the textbook scheme, its role in supporting teachers' use of representation is considered integral to this study.

1.3 Research Questions

Bearing these issues in mind, the primary aims of this study are firstly to make contributions to the body of knowledge about teachers' use of representations and to use this to influence policy and practice in developing mastery approaches to teaching mathematics. In doing so, a second aim of the study is to enable teachers to better understand beliefs and knowledge about the nature of mathematics and mathematical objects and how these might relate to classroom practice and the recontextualisation of knowledge (Muis, 2004; Philipp, 2007; Duval, 2006; Sfard, 2000). These aims are important because previous studies have shown that using multiple representations in specific ways can lead to improvements in pupil performance (Sowell, 1989; Meira, 1995; Cramer, Post and delMas, 2002; Rau et al., 2009; Carbonneau, Marley and Selig, 2013; Petersen and McNeil, 2013). However, little is known about what influences teachers' use of representations. Finally, a third aim of this study is to contribute more broadly to understanding the knowledge practices of teachers through applying Legitimation Code Theory. In this way, this study will be able to generalise to theory extending the explanatory power beyond the specific context of the study itself.

Bearing these aims in mind, the question that my study seeks to answer is:

How do teachers' mathematical beliefs and knowledge influence their use of representations in the process of negotiating the mathematical meaning of fractions?

To fully answer this question, there are also four related questions that help focus the study. These questions are designed to direct the research process towards key areas that will help answer the over-arching research question above. These will be used to structure the ensuing chapter:

1. *How can we effectively understand teachers' beliefs and knowledge about the nature of mathematics and mathematics education?*
2. *How can we effectively understand how mathematical representations are used by teachers to communicate mathematical meaning in school maths lessons?*
3. *How does a textbook scheme influence teachers' beliefs and knowledge, and use of representations?*
4. *How can we explain the relationship between teacher beliefs and knowledge and the use of mathematical representations in the classroom?*

The following chapter will provide an overview of the pertinent literature, providing a critical rationale for the above research questions and identifying where previous studies have not provided convincing answers.

2 Literature Review

The literature review for this study took an iterative approach. At first, the initial research focus was set out and a key word search relating to mathematics specific teacher beliefs, knowledge, and the use of representations was undertaken on the following databases: Academic Search Complete, Education Source, JSTOR, Taylor and Francis Online, ProQuest Education and Google Scholar. Through an initial review of this literature, the focus for the research was then developed to become more precise and this then led to further key word searches. This was primarily due to the wide variety of terms used by different authors to describe similar ideas and, upon reviewing the results of the first wave of searches, it was clear that the key words being used would need broadening so that all the available literature was covered. The following review of the literature is not exhaustive, instead it presents the most pertinent findings from these searches and outlines the theoretical and empirical issues that relate to the key areas of the philosophy of mathematics, teachers' beliefs and knowledge about mathematics teaching, the use of representations in mathematics teaching, and the role of textbooks. Throughout this chapter, I will demonstrate how previous research and theory has thus far failed to develop and apply a sufficient theoretical and analytical framework for the study of teacher beliefs and knowledge and how they relate to practice. This will provide a rationale for my study that will then lead to the Theoretical Framework and Methodology chapters where I will outline my new approach to this area of research. In particular, the complexity of this phenomenon means that it is more productive at this stage to design a study focussing upon one teacher in depth so that the intricacies of this area can be more fully understood and provide a better foundation for future research.

2.1 The Nature of Mathematics

Because my study is focussed upon teacher beliefs and knowledge, it is important to outline some of the broader philosophical arguments about the nature of mathematics and how they relate to school maths. In particular, this section will provide important details that can be used within my study when analysing and explaining teachers' mathematics related beliefs.

The nature of mathematics is contested and the varying beliefs about its existence are an important element of this study. Although there are many nuanced differences between individual philosophical approaches to the nature of mathematics, one of the key issues of importance is that of certainty (Hersh, 1999). Tracing back through the history of mathematics to the times of Plato and Euclid, it appears to have been a taken for granted fact that mathematical knowledge was certain and infallible (Charalampous, 2016). However, opinions about this have changed over time as the nature of the subject has been put under increasing levels of scrutiny (Hersh, 1999; Grattan-Guinness, 2000; Shapiro, 2005). Broadly speaking, the varying beliefs about the nature of mathematics might be considered to fall on a spectrum between two main categories: fallibilist or absolutist beliefs (Ernest, 1991).

Ernest (1991: 7) describes an 'absolutist' philosophical view of mathematics:

> The absolutist view of mathematical knowledge is that it consists of certain and unchallengeable truths. According to this view, mathematical knowledge is made up of absolute truths, and represents the unique realm of certain knowledge…

Within this broad stroke approach to describing philosophical viewpoints, it is important to acknowledge that this could be considered as one end of a spectrum of beliefs that do include significant differences. At the most extreme absolutist end of the spectrum would be 'platonism'. Platonism, named after the mathematical beliefs of Greek philosopher Plato, sets out mathematics as consisting of a set of ultimate truths that exist externally to the human race (Hersh, 1999; Charalampous, 2016). Such truths are not accessible by our senses, rather they are only accessible through our ability to reason. Differing significantly from this and perhaps less extreme, Ernest (1991) also considers ‘formalism’ to be closer to the absolutist end of the spectrum. Formalism, having developed traction through countering Euclid’s prominence of geometry in favour of arithmetic, is significantly different from platonism. Instead of seeing mathematics as absolute because of a meta-physical existence of some sort, formalists approach mathematics as a syntactical game where the subject matter is the symbolic system itself (Detlefsen, 2005; Charalampous, 2016). Within formalism, mathematics may still be considered absolute in nature but not because of something that exists in the universe, rather, because it is like a giant game with a strict set of rules that must be adhered to (Ernest, 1991; Hersh, 1999).

At the other end of the spectrum, Ernest (1991) contrasts the varying degrees of absolutist beliefs with what he terms ‘fallibilist’ beliefs. Fallibilist beliefs suggest that mathematics is fallible in nature and subject to historical change and human error (ibid., 1991). However, like the absolutist classification, within fallibilism there are also varying degrees of strength. At perhaps its most extreme there is the school of ‘humanism’ that has been significantly influenced by the work of Ludwig Wittgenstein (Wittgenstein and Anscombe, 1968). Humanism proposes that mathematics is a purely social construct where meaning is dependent on socially constructed language rules. Within this philosophy of mathematics, the correct answer to a question like ‘2 + 2’ is only ‘4’ because this has been decided within our society and there would be nothing to stop a different social group in deciding that ‘2 + 2’ is actually ‘5’ if they found that to fit better with their language and societal demands; mathematics is created and used in different ways by different societies (Wittgenstein and Anscombe, 1968; Hersh, 1999). However, this would be at the more extreme end of the fallibilist spectrum and others have proposed a more subtle, ‘quasi empiricist’ approach (Putnam, 1975; Lakatos, 1976; Ernest, 1991). This approach is summarised well by George Polya (1957: xxxiii) who describes it as “mathematics in the making”, advocating a view of mathematical knowledge that is created through human activity. As Polya’s quote suggests, within this approach, mathematics is still considered to be a social construct, but a method of mathematical knowledge creation that is more rigorous and somewhat akin to that of the sciences is promoted (Lakatos, 1976; Charalampous, 2016). In his seminal book ‘Proofs and Refutations’, Lakatos (1976) describes this view of mathematics through an imagined classroom discussion where mathematical meaning is generated through testing hypotheses with the view of developing a theorem. Within the quasi-empiricist approach, it is also important to highlight the role of pragmatism: proposed mathematical methods, or claims, should be useful and contribute to the advance of the subject in order to be accepted by the mathematical community (Putnam, 1975). This again highlights the importance placed on society in the creation of mathematical knowledge. Therefore, fallibilist beliefs are very much centred around human activity and are susceptible to historical change and human error, as Lakatos puts it “Mathematical activity is human activity” (Lakatos, 1976: 146).

This quasi-empiricist approach also appears to relate strongly to a broader, more recently developed and more domain general, social realist approach to knowledge. Within Social Realism knowledge is seen as fallible and socially constructed whilst still maintaining a sense of reality in that it has effects on society and carries intrinsic value (Moore and Young, 2009; Maton, 2014; Lilliedahl, 2015). From this viewpoint, knowledge is “emergent from but irreducible to the practices and contexts of its production” (Maton and Moore, 2009: p.5). In fact, when prominent social realist Michael Young (2013: 107) describes knowledge as “always fallible and open to challenge”, there appears to be a direct link to Ernest’s (1991) fallibilism, in particular, the quasi-empirical approach to mathematics. Within mathematics this might be demonstrated by our number system. When dealing with relatively small numbers, it is possible to find these numbers represented in concrete ways within our world (for example, I could show ‘three’ by getting three stones) – it is possible to see where this knowledge has emerged from throughout history. However, as soon as we extend way beyond small numbers into the realms of very large numbers, it is not possible to reduce these to any meaningful real-life context. For example, take the very large number googol, which can be represented as 10^{100} (ten to the power of one hundred). Not only is this number so large that it is very difficult to even contemplate, but it is also so large that there are perhaps not enough atoms in planet earth to be able to print it out in entirety on paper. Despite this, it is a number that can be discussed, and representations can be used to negotiate its meaning – it is a piece of knowledge with intrinsic value. This begins to demonstrate how mathematical knowledge is based upon social practices and contexts yet also irreducible to these as it can be generalised beyond them.

Such an approach to the creation of mathematical meaning has been advocated by many and is perhaps now the availing belief within the educational literature, even if it is not always explicitly acknowledged (Muis, 2004; Liljedahl, 2008; Hudson, Henderson and Hudson, 2015; Sun, 2015; Boaler, 2016). Nevertheless, just because it prevails within the literature, this does not mean it prevails within society and amongst teachers. It has been highlighted by some that, within our Western culture, mathematics is generally held to be absolute in nature and that this cultural attitude is also mirrored in much of the mathematics teaching that goes on in schools, even to the extent that some believe mathematics to be only for a chosen few who have the right sort of personal attributes (Boaler, 2016). This relates to Bernstein's (2000) knowledge recontextualisation, where common cultural beliefs appear to have influenced the recontextualisation of mathematics into 'school maths' that some consider as suffering from epistemic weakness (Henderson and Hudson, 2015). The suggestion is that when school maths is taught in this way, the subject of mathematics is presented as absolute and there is an emphasis on rule following, rigid answers to problems, and teacher exposition (ibid.., 2015).

In addition to considering the epistemological nature of mathematics, because this study is also focussed upon the representations of mathematical ideas, it is important to consider the ontological nature of the subject as well. The question needs to be asked, what are the objects of mathematical knowledge and what are they like?

In much the same way as the nature of mathematics, the ontological status of mathematical objects is also contested (Hersh, 1999; Cobb, Yackel and Wood, 1992; Sfard, 2000; Iori, 2017). Because of this, they are a slippery concept and difficult to define. However, in broad terms, a mathematical object is something that can be formally defined in mathematics, where there is a shared understanding of this definition within the mathematical community. Some objects are made up of smaller objects, for example everything from the clearly defined part of mathematics known as 'arithmetic' through to an expression such as '3 x 4', and even the number '3' itself, could all be considered as mathematical objects. Somewhat mirroring Ernest's (1991) absolutist category, one argument would be to see mathematical objects as having a real existence of sorts, what Sfard (2000: 43) refers to as an "objectivist", or "realist", stance. In contrast to this, and somewhat mirroring fallibilist beliefs, another approach is to argue that mathematical objects exist as a combination of social and cognitive activity, and are therefore subject to human objectivity, relying on communication within communities so a shared understanding is developed (Cobb, Yackel and Wood, 1992; Radford, 2006; Godino and Font, 2010). Either way, it is clear that mathematical knowledge objects lack ostensive referents, and this poses important questions that will be addressed in a subsequent section on representation (section 2.3). At this point, it is important to highlight that, much like fallibilist beliefs, the socially and cognitively situated nature of mathematical objects appears to be the most popular stance within the education literature. Again, drawing upon a social realist perspective of knowledge (Maton, 2014), despite the lack of ostensive referents, mathematical objects may be seen to be real in that they exist as a set of shared understandings that consequently exert their own influence and effects upon society. To put it simply, across most of the world when I use the number 'three' in dialogue (language dependent), others will be able to understand what I mean. The number 3 as a mathematical object has value within itself rather than being completely open to interpretation, yet it can still hold multiple meanings (figure 1) and is therefore equally, not absolute in nature.

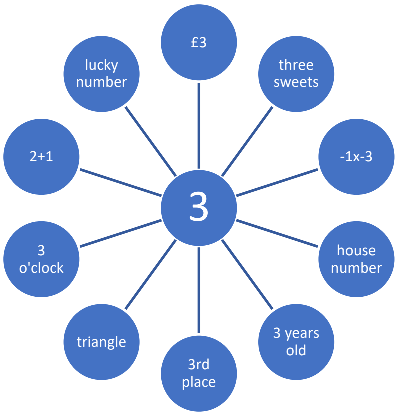


Figure 1 - An example of some possible meanings of the number '3'.

Bearing in mind these philosophical arguments about the nature of mathematics and mathematical objects, it follows that the concepts of 'mathematical beliefs' and 'mathematical knowledge' must also be interrogated in order to better understand how this relates to the focus of my research.

2.2 Believing and Knowing Mathematics

Teacher beliefs and knowledge about mathematics have been investigated by many in attempts to try and explain the way in which they influence mathematics education (Shulman, 1986; Thompson, 1992; Ma, 1999; Hill, Schilling and Ball, 2004; Muis, 2004; Philipp, 2007). Nevertheless, there appear to be two important issues that need to be addressed. Firstly, the term 'belief' itself is often ill-defined and used in different ways to mean different things within the literature (Pajares, 1992). Secondly, differentiating between knowledge and beliefs appears to be a highly imprecise activity. Schoenfeld (1992), who made significant contributions to the literature on mathematical beliefs, broadly describes mathematical beliefs as a person's mathematical worldview through which mathematical behaviour is determined. Philipp (2007: 259), broadening this, provides a useful definition:

> **Beliefs** – Psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes.

Within this definition, the second important issue is raised. Here, Phillipp (2007) is summarising a range of literature and concludes that to differentiate between beliefs and knowledge it is the conviction, or strength, with which something is held that is the defining characteristic. However, upon further scrutiny this in itself is very difficult to determine in practical terms. To illustrate this, take the mathematical notion of 'order of operations' (table 1). To many, this is considered to be mathematical knowledge. Take the following example:

$$\mathbf{64 \div 4(2 - 6)}$$

A pupil who has been taught to use the order of operations rule, and uses it rigidly, will calculate the brackets first, so we do 2 – 6 = -4 and then we would do division next ($64 \div 4 = 16$) and then finally the multiplication (16 x -4). This leaves the pupil with a final answer of -64. However, it is also acceptable to re-write the same question in this way:

$$\frac{\mathbf{64}}{\mathbf{4(2-6)}}$$

In this case, because the division is written using a vinculum instead of the obelus (÷) symbol, what the pupil does first changes. They must now multiply the 4 with the -4 first leaving us with a final answer of -4. Does this mean the pupil's knowledge of order of operations is now no longer correct? Or, alternatively, that this one calculation has two possible answers? In this case, some people may think they 'know' the order of operations that must be followed whereas, this appears to just be a belief in the way things should be done in certain circumstances; which one do you *believe* to be correct and how would you know? Some people may state that ‘In order to calculate a problem with more than one operation, we must follow the order of operations rule’ whereas others may state that ‘In order to calculate a problem with more than one operation, usually we should follow the order of operations rule but there are exceptions to this’. This example highlights the blurred line between knowledge and beliefs, and it is possible that what is commonly considered to be mathematical knowledge by some is actually held more or less strongly as a belief by others. Therefore, the differentiation between knowledge and beliefs is a highly personalised issue where distinctions seem very difficult to make. For the purpose of my study, it is arguably most pertinent to consider firstly what teachers and pupils *think* about their beliefs and knowledge, sometimes referred to as ‘personal epistemology’ (Hofer, 2001), and compare this to what actions can be observed, rather than attempting to precisely define what is a belief and what is knowledge. Secondly, because beliefs and knowledge are so intricately linked, it is perhaps more useful to consider how the two can be studied together rather than trying to separate them. This stance will be further elaborated through analysis of the empirical research in subsequent sections.

Table 1 - The Commonly Taught Order of Operations Rule

B	***Brackets***	10 x (4 + 2) – 30 = 10 x 6 - 30
I	***Indices***	4^2 + 6 = 16 + 6
D	***Division***	10 + 6 ÷ 2 = 10 + 3
M	***Multiplication***	10 – 4 x 2 = 10 - 8
A	***Addition***	10 x 4 + 7 = 40 + 7
S	***Subtraction***	10 x 2 – 15 = 20 - 15

Emanating from the seminal Perry Scheme, the idea of personal epistemology is that people have beliefs about their own knowledge, thus approaching personal philosophical beliefs from a psychological perspective (Perry, 1970; Hofer, 2001). The scheme itself was derived from research with American college students and sought to map out different stages of epistemological development that students go through (Perry, 1970). Students at the early stages of this developmental scheme were described as having a ‘dualistic’ perspective of their own knowledge that is characterized by an absolute view of knowledge and a belief that a teacher's job is to communicate these absolute truths to students (ibid., 1970). This work was developed further by Schommer (1990) who investigated the idea of personal epistemological beliefs in the context of reading and described epistemological beliefs as ranging from 'naive' to 'sophisticated'. In her model, naive beliefs were similar to Perry's (1970) dualistic phase where a person believes knowledge to be absolute. In contrast, a person with so-called sophisticated beliefs held their own knowledge in a more tentative manner (Schommer, 1990). These somewhat mirror the philosophical views of mathematics proposed by Ernest (1991) with naive beliefs being similar to an absolutist view and sophisticated beliefs being similar to a fallibilist view of mathematics. Arguably, the wording chosen by Schommer (1990) to describe these different beliefs highlights the point that fallibilist beliefs seem to be most popular within educational literature, with the terms 'sophisticated' and 'naive' carrying an element of value judgement. As Muis (2004) points out, describing a certain epistemological perspective as 'naïve' suggests a bias towards one viewpoint over another (i.e. that knowledge is tentative). Instead, Muis (2004) proposes using the terms 'availing' and 'nonavailing' instead of 'sophisticated' and 'naive'. In order to clarify these terms, Muis (2004: 6) states that:

> Availing beliefs are associated with better learning outcomes, and nonavailing beliefs have no influence on learning outcomes or negatively influence learning outcomes.

This alternative language brings the discussion back to empirical ground and prompts questions about what type of mathematical beliefs and knowledge are associated with better learning outcomes. The following sections will firstly focus on defining mathematical beliefs and reviewing the empirical research, followed by discussing mathematical knowledge, leading to a discussion of a theoretical model for studying beliefs and knowledge together.

2.2.1 Mathematical Beliefs

The study of beliefs in relation to mathematics education has received a significant amount of attention within the literature however there is also a very broad range of interpretations as to what this actually refers to (Thompson, 1992; Phillipp, 2007). Do we take it to mean philosophical beliefs about the nature of mathematics, as discussed previously, or even knowledge of these beliefs? Or do we take it to mean beliefs about how people learn and go about doing mathematics (beliefs about practice, efficacy and intelligence)? Within the literature it appears as though the term has been used in a variety of ways with some using it to refer to beliefs about mathematical intelligence (Sun, 2015; Bonne and Johnston, 2016), others using it to refer to beliefs about how mathematics should be taught or learnt (Kloosterman and Cougan, 1994; House, 2006; Correa et al., 2008), some who use it to mean beliefs about the nature of mathematics (Schoenfeld, 1989; Adnan, Zakaria and Maat, 2012) and some who incorporate a number of these (Perry, Howard and Tracey, 1999; Paolucci, 2015; Boyd and Ash, 2018a). In an attempt to create a useable theoretical framework that summarises the general approaches to distinguishing mathematical beliefs, De Corte, Op't Eynde and Verschaffel (2004: 303) suggest the following three categories of mathematical beliefs:

1. Beliefs about mathematics education.
2. Beliefs about the self in relation to mathematics.
3. Beliefs about the social context, i.e., the context of mathematical learning and problem solving.

Although presented neatly in three categories here, it is a highly complex area of study with each one consisting of a number of sub-categories. Ensuring a detailed understanding of these will be of central importance to this study therefore, in order to provide clarity, these have been distilled into a table (table 2) that provides further description of the framework.

Table 2 - Beliefs About Mathematics Framework (Adapted from De Corte, Op't Eynde and Verschaffel, 2004)

Mathematical Belief Category	**Sub-categories**
Beliefs about mathematics education	• Beliefs about the nature of mathematics • Beliefs about how to learn mathematics and apply it (E.g. when problem solving) • Beliefs about the teaching of mathematics
Beliefs about the self in relation to mathematics	• Goal orientation (E.g. whether a person is seeking to understand something or to simply get the correct answer) • Values related to the mathematical content (E.g. whether a person believes it is/isn't important to learn the mathematical content) • Beliefs related to control over learning (E.g. Does a person believe that learning the content is within their control and if they learn in adequate ways then they will succeed) • Self-efficacy beliefs (E.g. does a person believe they are capable of learning the mathematics)
Beliefs about the social context	• Beliefs about what is valued by others in the social context (E.g. Does a person believe that others will negatively judge them for stating an incorrect answer) • Beliefs about the expectations of the teacher (E.g. Do the pupils believe that the teacher expects them to discuss ideas with one another) • Pupils' beliefs about what teachers should do to teach mathematics effectively (E.g. do pupils believe that a good teacher will tell them what to do)

Within this framework the three categories are interconnected. For example, within the domain 'beliefs about mathematics education', beliefs about the nature of mathematics are a component (ibid., 2004). However, as Sun (2015) has shown, what pupils believe about the nature of mathematics is likely to have an impact on what they believe about their self-efficacy in relation to mathematics, which in turn is influenced by their teacher's classroom practice. Beliefs related to these are included in both the first and second domains ('beliefs about mathematics education' and 'beliefs about the self in relation to mathematics'). This aligns with others who have suggested that beliefs exist within belief *systems* that are interconnected (Thompson, 1992; Leatham, 2006). Based on the idea that a belief does not exist in isolation from other beliefs but instead they exist within mental structures, with some beliefs acting as the basis of others, belief systems are considered to be made up of primary and derivative beliefs (ibid., 1992). For example, if a person holds a primary belief that mathematics is an unchallengeable body of knowledge then this may well lead to a derivative belief that learning mathematics predominantly involves memorizing rules and facts. However, it is also important to consider how strongly beliefs are held as this speaks to their susceptibility to change. Thompson (1992) suggests that it is not necessarily the case that primary beliefs are held more strongly than derivative beliefs and, in some cases, it may be the other way around. In the previous example, although believing that learning mathematics involves memorizing rules and procedures was a derivative belief, it may actually be the case that this is held more strongly (and therefore is harder to change) than beliefs about the nature of mathematics. Arguably this is especially likely to be the case for many primary based teachers in England who tend not to be subject specialists and are therefore unlikely to have considered the ontological or philosophical nature of the various areas of mathematics they teach.

The concept of belief systems appears to be a complex one and it is hard to separate one sub-category of mathematical beliefs from another. In addition to this, despite there being some evidence of mathematical beliefs being domain specific (Löfström and Pursianinen, 2015), it may well be the case that non-mathematics related beliefs impact upon the way a teacher or pupil acts in a mathematics lesson. Somewhat in answer to this, Törner (2002) has proposed that beliefs may be different depending on their level of globality and that beliefs may range from being about very broad, global topics through to very precise teaching activities or even specific instances in time.

2.2.2 Teachers' Mathematical Beliefs

Previous research has shown that, where explicit efforts are made to do so, teachers' classroom practice can positively impact pupils' beliefs about mathematics which, in turn, can positively impact learning outcomes (Kloosterman and Cougan, 1994; Hofer, 1999; Muis, 2004; Bonne and Johnston, 2016). This poses the question as to whether there is a link between teachers' own mathematical beliefs and their classroom practice.

In much of the literature on teachers' beliefs about mathematics there appears to be the assumption that certain teacher beliefs pertain to particular teaching practices. For example, when Sun (2015) refers to teachers' beliefs about the nature of mathematics she uses the terms 'multi-dimensional' and 'one-dimensional', however in her definition of these, she refers to beliefs about the *teaching* of mathematics and not the nature of mathematics itself. This blending of philosophical beliefs and beliefs about instructional practices has been repeated by others (for example Liljedahl, 2008) and appears to stem from a common practice which is to "derive instructional prescriptions directly from background theoretical perspectives" (Cobb, 2007: 3). As an example, it might be assumed that where a teacher has something akin to a fallibilist view of mathematics then their classroom practice will present the subject in an open way with problem solving and creativity at its heart. This may be taken even further; one might assume that mathematics presented in this way will then support the development of fallibilist, availing beliefs amongst pupils, however this is still an assumption. Thompson (1992) has shown that this assumption is a common one amongst teachers also, emphasizing that teachers' beliefs about the nature of mathematics are often related to their beliefs about how to teach it. These assumptions may seem like common sense; however, it has been argued strongly by others that this mixing up of subject nature beliefs and teaching practices is flawed and at times even damaging (Kirschner, 2009). Therefore, it is important to first consider the existing evidence about the relationship between teachers' beliefs about the nature of mathematics, beliefs about the teaching of mathematics, and actual classroom practice.

Investigating these issues, Erikson (1993) employed a case study approach using a variety of qualitative and quantitative methods. She focused on two middle school teachers in the USA attempting to find out more about the relationship between teacher beliefs about the nature of mathematics, classroom practice and pupil beliefs. The first teacher (Teacher A) began the study by asserting a belief that mathematics was all about basic skills and was a tool to solve number related problems and, by the end of the study this had changed to a belief that "mathematics is a universal language that explains and describes why anything works" (ibid., 1993: 10). In addition to this, Teacher A also stated that problem solving played a large role in his teaching practice. Despite this apparent shift in mathematics related beliefs, Teacher A's classroom practice actually changed very little throughout the study. Specifically, at the end of the study his classroom practice is described as "teacher-led" where there is little pupil talk and the majority of time is spent either as a whole class with the teacher leading discussion, or quietly completing independent work (ibid., 1993: 13). It is also interesting to note that, in Teacher A's class, pupils held a belief that mathematics was primarily a tool to solve problems, that memorization of facts and procedures was central, and that the teacher must show you how to solve a problem before you can do it independently. This suggests that it is possible for teacher beliefs to shift quite significantly, yet for classroom practice to change very little. Erikson (1993) concludes that because Teacher A's classroom environment remained 'traditional' in nature, the shift in beliefs had very little impact. However, it is interesting to note that, although Teacher A's beliefs did shift to viewing mathematics as a broader subject (a universal language), the way views are presented in the paper still seem relatively absolutist in nature. Specifically, the views expressed at the start of the study (that mathematics was essentially just a tool) seem almost Formalistic (mathematics is like a game with a clear and useful set of rules) and the move to seeing mathematics as a universal language seems to indicate a shift to a more Platonist view, both of which are still absolutist in nature (Ernest, 1991). In comparison, the second teacher in the study (Teacher B) consistently emphasized the importance of communication in mathematics stating, "the usage of mathematics is the reason for its existence" (ibid., 1993: 18). Teacher B also emphasized that problem solving, and cooperative working was a central aspect of her classroom practice. In contrast to Teacher A, Teacher B's observed classroom practice did seem to match her beliefs, with observed examples that include an emphasis on developing conceptual understanding and the communication of mathematical thinking through collaborative discussion (ibid., 1993). The pupils in Teacher B's class also held beliefs about mathematics that emphasized it as a way of thinking about things, something they could do independently, and that common sense was more important than learned rules. Again, it is interesting to note that Teacher B's views of mathematics emphasise the human aspect of it, most notably that it is a subject that exists because it is used. This seems to suggest a more fallibilist position as there is a focus upon mathematics as socially constructed.

Similarly, in an American Elementary school, Raymond (1997) studied the interplay between teachers' beliefs about the nature of mathematics, about learning mathematics, about pedagogy and their actual classroom practice. Using a mixture of interviews, classroom observation and sampling of lesson planning, Raymond (1997) focuses primarily on the case of one individual teacher given the name 'Joanna'. In the study, Joanna was found to have very 'traditional' beliefs about the nature of mathematics that had been derived from her own school experiences of the subject. These beliefs include viewing mathematics as fixed, absolute, and certain which suggests an absolutist stance. Nevertheless, Joanna also expressed what were described as 'non-traditional' views about learning mathematics and pedagogy. Namely, that collaboration and discovering mathematics and hands on working using manipulatives were important, and that good teachers took time to investigate multiple ways to solve problems. Such findings seem to challenge the assumption about teachers' beliefs discussed earlier, however it is only when we hear about Joanna's actual classroom practice that we see the full picture. Raymond (1997: 565) describes her teaching environment as one where "she orchestrated the presentation of topics while students quietly looked on". Alongside this, almost no use of manipulatives or collaborative working was observed. This poses the question as to why Joanna expressed strong beliefs in so called non-traditional teaching practices if she did not actually use them? Raymond (1997) concludes that other competing factors are likely to have led to this inconsistency such as worries about pupil behaviour, time constraints, lack of resources and worries about school testing. This suggests that the enacting of beliefs about pedagogy is complex and closely related to other contextual factors created by the system within which teachers work. Nevertheless, as Raymond (1997) suggests, the case of Joanna seems to suggest that beliefs about the nature of mathematics may have a stronger influence on classroom practice than beliefs about pedagogy and learning mathematics.

The apparent inconsistencies relating to teachers' mathematical beliefs and actual teaching practices within the previous two studies do not seem to be anomalies (De Corte, Op't Eynde and Verschaffel, 2004; Philipp, 2007). However, as Philipp (2007) suggests, it may well be the case that these *appear* to be inconsistencies to researchers yet may well *not* be inconsistencies within the minds of the teachers being studied. It is easy to see a difference between what someone says they believe and then how they act, and to explain it away by calling it an inconsistency, when in reality it is not an inconsistency at all. In addition to this, if beliefs are considered to exist within belief systems (Thompson, 1991; Leatham, 2006), it might well be the case that a teacher holds both seemingly inconsistent beliefs within a system that means in certain scenarios one belief outweighs another. That is to say, beliefs may well be context specific – in any given situation some beliefs may influence behaviour more strongly than others and this is likely to be dependent on the belief system held (Törner, 2002; Leatham, 2006; Philipp, 2007). Philipp (2007) argues that instead of explaining away such as issues by calling them inconsistencies, researchers could assume that inconsistencies are not present which is likely to lead to a better understanding of belief systems. It is important to highlight here that the suggestion is not that inconsistent beliefs do not exist; it is to prompt research designs that strive to better understand belief systems (ibid., 2007). Perhaps one reason for this gap in the research is that there has been a lack of any theoretical framework which enables a better understanding of this phenomenon. This is one of the reasons why, within this study, it is not only important to consider the context of a belief within a belief system, but also how that belief system fits together with a person's knowledge, and what they believe about their own knowledge. This is especially important given the blurred lines between what is held as mathematical knowledge and what is held as a mathematical belief. For example, in the study by Erikson (1993), one of the teachers whose beliefs shifted but classroom practice did not, may have had insufficient (or believed she had insufficient) knowledge of pedagogical strategies to turn those beliefs into a reality and this may have been an explanation as to the lack of change within her classroom. Had the study included teacher knowledge as well, therefore developing a more multi-dimensional theoretical framework, they may have been able to understand the phenomena they were investigating in a deeper way rather than explaining it away as an inconsistency. Therefore, I believe that it is more relevant and useful for empirical research aiming to influence policy and practice to consider belief *and knowledge* systems rather than separating the two, and an adequate theoretical framework that enables this will be required. This leads onto an important discussion about the research into teachers' mathematical knowledge and how this might relate to beliefs.

2.2.3 Teachers' Mathematical Knowledge

Considering the fine line between beliefs and knowledge discussed earlier (Philipp, 2007), it is possible to conceive that one person's mathematical knowledge (what they believe to know in certainty) might be another's belief (something they believe but know not to be certain). It is important to note here that subject knowledge does not simply refer to 'being able to do the mathematics'. It has been noted by some (Schulman, 1986; Ma, 1999; Ball, Thames and Phelps, 2008) that, in order to teach mathematics, a particularly specialised body of knowledge is required; what Ma (1999: 118) describes as a "profound understanding of fundamental mathematics". This involves not only being able to do the mathematics but also understanding the developmental way in which mathematics is learned and how teaching can support this. Perhaps surprisingly, studies have found that simply having higher levels of mathematics qualifications does not necessarily lead to better knowledge for teaching mathematics (Askew et al., 1997; Ma, 1999).

Much of the work investigating teacher subject knowledge stems from the work of Shulman (1986) who categorized the type of knowledge teachers needed into four main areas: subject matter content knowledge, curricular knowledge, general pedagogical knowledge and pedagogical content knowledge. The first three of these are fairly self-explanatory and refer to the knowledge teachers have of their subject, of their curriculum and of general pedagogy, however it is the last of the four that has received the majority of focus within mathematics research. Pedagogical content knowledge (PCK) refers to the professional knowledge a teacher has of the way in which specific aspects of a subject can be taught most effectively (ibid., 1986). Building on this model, others have developed the Mathematical Knowledge for Teaching (MKT) framework that particularly focuses on subject matter knowledge (herein referred to as Content Knowledge [CK]) and PCK (Ball, Thames and Phelps, 2008; Hill, Ball and Schilling, 2008). The framework describes this sort of knowledge by splitting each one up into sub-categories that have been summarized in the table below (table 3).

Table 3 - The Mathematical Knowledge for Teaching Framework (Adapted from Hill, Ball and Schilling, 2008)

Mathematical Content Knowledge	Common content knowledge	The sort of everyday mathematics that you might expect an average adult to have. For example, calculating whether you can afford to buy something.
	Specialised content knowledge	The sort of knowledge that teachers have regarding the mathematics they teach to pupils, and how pupils go about learning it. For example, being able to identify what a pupil has done wrong in the following calculation: 49 × 25 225 100 325
	Horizon content knowledge	Knowing where the mathematics being taught fits in to the 'bigger picture'. What the purpose of the current thing being taught is for the mathematics that will be done in the future. For example, knowing that pupils learning associated division facts along with times tables because, in the future, this will be useful when doing long division.
Pedagogical Knowledge	Knowledge of content and pupils	Knowing what areas of strength and weakness the pupils in a class have. Knowing common errors made by pupils at certain ages. For example, knowing that pupils aged 4 and 5 often count objects more than once when counting a set.
	Knowledge of content and teaching	Knowing the mathematics to be taught and its associated pedagogy. For example, being aware that using physical manipulatives can help pupils conceptually understand the manipulation of algebraic equations.
	Knowledge of curriculum	Knowing the mathematics content that has to be taught as part of the school curriculum. For example, knowing that by the end of year 2 (7 years) pupils should be fluent in their 2, 5 and 10 times tables.

Using a research instrument developed to measure a teacher's level of MKT (Hill, Schilling and Ball, 2004), there has been a significant amount of research relating to the issues of developing teachers' level of MKT, the link between MKT and pupil performance, and the link between MKT and teaching practices (Hoover et al., 2016). Most significantly to this study, the final area of study mentioned here is one in need of further investigation. Although it seems clear that MKT does influence classroom practice, the extent or way in which it does is still unknown (ibid., 2016).

Using a multiple case study approach, Hill et al. (2008) attempted to unpick the relationship between MKT and classroom practice in significant detail. The study utilised the MKT measure instrument, videos of lessons, post-lesson debriefs and teacher interviews. Within the study they identified five teachers who illuminated different elements of the relationship between MKT and classroom practice. Most notably, it was not the teacher with the highest MKT score that demonstrated the most effective classroom practice and the authors cite mathematical beliefs as one possible reason for this (although these were not measured in the study). In particular it was the elements of "richness of mathematics, classroom work being connected to mathematics, and responding to students appropriately" that were most variable amongst teachers with a high level of MKT (Hill et al., 2008: 497). Such findings highlight how mathematical knowledge is important but must be considered alongside mathematical beliefs.

Adding weight to this proposal and attempting to better understand the relationship between subject knowledge and beliefs, Sleep and Eskelson (2012) compared the implementation of mathematics curriculum reform materials in the United States between two teachers: one with high levels of mathematical knowledge for teaching (MKT) and another with average levels. They discovered that, when both teachers were delivering the same lessons using the same curriculum materials, there was significant difference in their delivery. Despite the expectation that the teacher with higher levels of MKT would deliver the lesson more effectively, they discovered that the teacher with lower levels of MKT had a higher quality of instruction. They determined that, despite the difference in MKT levels, the teacher with lower MKT levels had a broader belief about the foundations of mathematics that appear to be more aligned with fallibilist beliefs about the nature of the subject. They suggest that high quality curriculum reform materials and good MKT are not enough to ensure high quality instruction and that teachers' beliefs about mathematics and mathematics instruction are an essential factor in mediating such reforms. Nevertheless, the authors themselves acknowledge that the teacher with a higher level of MKT scored most points on questions about mathematical procedures and did not do so well on non-routine mathematics. This suggests that the MKT measuring instrument used perhaps missed some important aspects regarding the teachers' understanding of concepts and their applications. This highlights a potential flaw with such research instruments that attempt to make generalised statements about such a complex and nuanced issue. This, again, raises the question of whether a different framework that allows for beliefs and knowledge to be studied is required.

Both aforementioned studies (Hill et al., 2008; Sleep and Eskelson, 2012) appear to reiterate the important connection between mathematical beliefs and mathematical knowledge when attempting to study classroom practice. However, it can be seen that the MKT model itself is not without problems from a theoretical position as well. Firstly, as has been suggested before, the relationship between knowledge and beliefs is a conceptually difficult one and it is arguably very difficult to define empirically. This suggests that a model of teachers' professional knowledge would be more useful if it acknowledged that beliefs and knowledge are tricky to differentiate from one another and must therefore be considered together. This is particularly the case for studies attempting to conduct pragmatic research that is aimed at furthering knowledge that can be directly applied to the practice of teaching and teacher education. Secondly, it appears as though the sub-categories presented within the MKT model are tricky to differentiate between in practical terms. For example, within the MKT model, specialised content knowledge is referred to as knowledge that is not needed for purposes other than teaching, however this appears very close to what might be considered by many as PCK and it poses the question as to whether having two very similar categories is even useful from a practical perspective (Petrou and Goulding, 2011). Such questions about definitions of sub-domains have been highlighted by a systematic review of the literature and continue to be an area of contention (Depaepe, Verschaffel and Kelchtermans, 2013).

Others have attempted to describe teachers' professional knowledge for mathematics teaching by building on Shulman's work but emphasizing the situated nature of knowledge. Within their model, Fennema and Franke (1992: 162) highlight similar aspects of knowledge to others (knowledge of mathematics content, knowledge of pedagogy and knowledge of pupils) but they centre this with what they term "context specific knowledge". Not dissimilar to those who argue that beliefs are context specific (Leatham, 2006; Philipp, 2007), this model suggests that a teachers' knowledge is dynamic and, when combined with beliefs in a classroom context, comes together to influence how a teacher acts (Fennema and Franke, 1992). Of particular relevance to this study, the notion of teacher "knowledge of mathematical representations" is given some thought within this model (ibid., 1992: 153). The authors point out that knowing how to represent mathematics so that pupils can understand it, is a key element of a teacher's mathematical knowledge. This is a particularly important area of subject knowledge to consider here as it brings to the fore a key aspect of this study: how teachers use and choose representations. In addition, it can be seen that this model makes some headway on drawing together knowledge and beliefs by acknowledging that "it is impossible to separate beliefs and knowledge", however it emphasizes issues surrounding teacher knowledge whilst leaving discussion about the beliefs element to others (ibid., 1992: 147). This approach does not seem uncommon and appears to have been repeated by others (Petrou and Goulding, 2011).

Despite this, there is a model that has received a small amount of attention within empirical studies that is worth considering (Kuntze, 2012; Dreher and Kuntze, 2015) (figure 2). This model of teachers' professional knowledge presented by Kuntze (2012) takes into account the fuzzy line between beliefs and knowledge by acknowledging multiple dimensions. Specifically, he acknowledges that each aspect of subject knowledge could either be a belief or knowledge depending on the individual and the context (ibid., 2012). In addition to this, the model also seems to take into account the notion of context specific beliefs and knowledge (Fennema and Franke, 1992; Leatham, 2006; Philipp, 2007) as well as different types of knowledge (Shulman, 1986). Building on the work presented by Törner, (2002), Kuntze's model suggests that some dimensions of subject knowledge might either be *global* (they are held generally about most situations), *content domain-specific* (e.g., beliefs and knowledge about the teaching of fractions), *related to particular content* (beliefs and knowledge about a particular mathematical activity) or *related to a specific instructional situation* (beliefs and knowledge within a real-life teaching situation). It is important to point out that, although the different dimensions are presented as discrete cells in the diagram, there is likely to be overlapping between each of them. This study will use Kuntze's (2012) model as a framework to support the design of teacher belief and knowledge tasks in order to facilitate data collection that includes a wide breadth of a particular teacher's beliefs and knowledge about fractions. It's role within the wider theoretical framework for my study will be discussed subsequently in section 3.3.

Figure 2 - Kuntze's (2012) theoretical model of teachers' professional knowledge

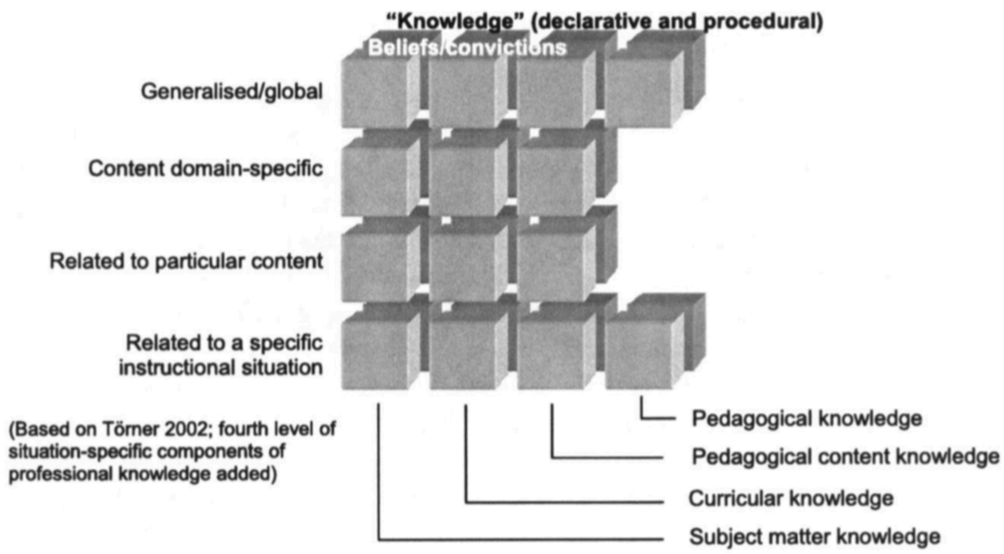


Figure 3 - Kuntze's (2012) theoretical model of teachers' professional knowledge

Overall, it appears as though there is general consensus that teacher beliefs about mathematics play an important role in influencing classroom practice, however the way in which this happens is an area in need of further research. There are several assumptions made within the literature about the link between teacher beliefs, classroom practice and pupil beliefs, which adds to this lack of clarity. It also seems that teachers' professional knowledge plays a significant role in influencing classroom practice but, because many models of teachers' professional knowledge do not take beliefs into account, the relationship between beliefs, knowledge and practice remains unclear. Because of this, more research is required to understand the precise ways in which teacher beliefs and knowledge actually impact classroom practice. In my study I aim to shed light on this issue by focusing on a specific aspect of classroom practice (use of representations) and it is therefore necessary to consider teacher knowledge and beliefs in relation to what is already known about using representations in school maths.

2.3 Representations in Mathematics

As has been discussed previously (section 2.1), the mathematical body of knowledge is built up of mathematical objects however, the nature of these objects is contested, and this means that questions regarding representation in mathematics are problematic. What are the representations of mathematics actually representing and how are they used to communicate mathematical meaning? There is a complex relationship between representation and the philosophical arguments about the nature of mathematics itself and, from the perspective of a school mathematics teacher, the current situation is neither clear nor desirable. If those in the business of constructing philosophical arguments about the nature of mathematics and mathematical objects struggle to agree, then what is to be expected of those in the tricky job of teaching? Should they be excluded from such arguments and encouraged to simply teach what they think they know? This is possibly a fairly accurate reflection of the current state of affairs throughout the English education system. However, as some have argued, this may lead to mathematics teaching that suffers from "low epistemic quality", meaning that the mathematics experienced in school is not an accurate enough reflection of the actual subject itself (Hudson, Henderson and Hudson, 2015: 377). This study aims to contribute to such arguments by focusing on mathematical objects, their representations (including beliefs and knowledge of these), and how these can support the process of learning mathematics, aiming to draw out these philosophical arguments in a way that is both clear and useful for teachers by connecting theory to practice. Helping make some headway towards this, some have attempted to define different types of mathematical objects and to describe systems for how these interact with one another through representation (Cobb, Yackel and Wood, 1992; Goldin, 1998; Duval, 2006). However, each of these approaches to mathematical objects and representations differ in nuanced ways and must be carefully analysed to provide a clear picture of the relationship between mathematical objects and their representations.

2.3.1 Internal and External Representations

Goldin (2002a) highlights the important role of representation when considering mathematical objects. He suggests that a representation is a configuration (visual, verbal, symbolic etc.) that represents another thing and that, in mathematics, it is the use of these that aids the development of mathematical meaning by the individual. Goldin, along with other proponents, argue that we should consider two different types of representation: external and internal (Goldin, 1998, 2002; Pape and Tchoshanov, 2001).

External representations refer to any type of representation of mathematical objects that can be physically experienced by others (through seeing, hearing, touching etc.) This includes spoken and written language, formal mathematical symbols, visual diagrams, physical equipment and mathematical configurations of things found in real life. These external representations are often established over time and may start out as individualistic and personalised but, through communication and collaboration, become entrenched and formalized in mathematics (Goldin, 2002a). For example, the use of the minus symbol to represent something like ‘-4’ has not always been common. In the seventeenth century no one would have recognized it or understood its meaning, yet it is now a central part of mathematics. According to Goldin and Janvier (1998) these representations may take the form of language systems, formal mathematical constructs (including symbolic systems) or physical situations (such as mathematical equipment or real-life scenarios that embody mathematical ideas). Within this theory, it is these representations that are used to communicate mathematics between people and that aid the development of mathematical meaning for the individual (Goldin and Shteingold, 2001; Goldin, 2002a).

The idea of internal representations refers to the fact that, in order to have created any external representation in the first place, a person must already have an internal mental representation of the thing that they are attempting to represent externally (Goldin and Shteingold, 2001; Goldin, 2002a; Pape and Tchoshanov, 2001). Such representations may include assignments of meaning to external representations or natural language and can be both cognitive and affective in nature (Goldin, 1998). For example, when a pupil goes to represent the number 23 with base ten blocks, they must already have an idea in their mind (an internal representation) of what they are showing with the blocks themselves and this might involve anything from a cognitive visualization of the number to an emotional feeling about it. This final point about a person’s affect towards a mathematical object is of particular interest because it is perhaps not intuitively what a teacher may consider to be an internal representation. As Goldin (2002b) highlights, a person’s affect towards a mathematical object is not auxiliary to cognition, it is in fact intertwined with it. When dealing with mathematical objects a person’s cognition and affect will be working together in a highly complex way to create a personal representation of it (ibid., 2002b). This is perhaps one reason why research has shown “mathematics anxiety” to have such a negative impact on learning (Carey et al., 2019: 6). As such, when considering representation of mathematical objects, personal affect towards them must be taken into consideration.

Considering these definitions of internal and external representations, it is speculated that a person's existing internal representation of a mathematical object is likely to influence the way in which they interpret an external representation and vice versa (Pape and Tchoshanov, 2001). The difficulty with this aspect of the theory is that a person's internal representation is not directly accessible to others, however it is argued that by closely observing how a person treats, uses, and creates external representations, we can gain insight into their internal representations (ibid., 2001).

It is arguable that this theory has become quite widespread amongst teachers due to its high level of plausibility. At first glance it seems fairly self-evident - that mathematical meaning is derived from a sort of 'back and forth' between internal and external representations, however it has been critiqued for being overly simplistic and leading to transmission style teaching practices (Cobb, Yackel and Wood, 1992; Godino and Font, 2010). That is not to say that there are not any useful aspects of the idea. More that it provides a useful starting point from which other theories can be used to interrogate the finer details of how mathematical objects and their representations interact with one another in the process of meaning making.

2.3.2 Types of Representation

Although the work by Goldin (1998, 2002a, 2002b) on internal and external representations does attempt to identify different types of representation, the main area of focus is on the interplay between internal and external and how this process occurs. The work of Duval (2006) focuses more closely upon the importance of types of external representation and how individuals use them to create meaning. Despite being influenced by absolutist mathematical philosopher Gottlob Frege (Hersh, 1999), Duval attempts to sidestep any questions regarding the ontological status of mathematical objects and claims that by focussing on representation as a tool for teaching, such arguments are not necessary in a pragmatic sense. The main premise behind his work is that mathematics is distinct from most other subjects because the objects that constitute the domain of mathematics cannot be found using perception or instruments of discovery (Duval, 2006). In other words, we cannot 'find' a mathematical object existing in real life in the same way we could find a table, a cake or any other physical thing. Others have disputed this, suggesting that with developments in modern molecular Chemistry, there are also areas of science that are equally inaccessible (Grosholz, 2007). Whilst this may be true, mathematics is significantly different because the whole subject is inaccessible in this way rather than just aspects of it, and this relates even to basic mathematics studied by young children. This arguably applies no matter what epistemological or ontological stance you take; whether mathematical objects are real abstract entities, or intersubjective social constructs, they are still not accessible via human senses other than through representation. Therefore, this study accepts Duval's proposal as a key issue when considering representation in mathematics. What this means in practice is that the only way it is possible to access and communicate about mathematical objects is by using representations of them. This poses a significant issue for mathematics educators: mathematical objects can only be accessed through representations, yet none of these representations are the mathematical object itself (Duval, 2006). Alongside this, questions have been raised as to whether multiple representations do actually represent the same object, or whether they just represent very similar objects (Sfard and Thompson, 1994). For example, do the two expressions '3+4' and '2+5' represent the same thing or are they representations of different things that are connected? The term 'expression' used in a mathematical sense conveys the idea that these are both expressing something about a certain object, and it is also possible to write them in this way '3+4 = 2+5', suggesting that they are equivalent. Thus, from perhaps a more mathematical perspective they are the same, yet in another sense they are different, they both show something distinct. In this study, whilst accepting the complexity of this situation, when mathematical expressions are equivalent, they will be referred to as representing the same thing. As all of this suggests, the job of the mathematics teacher is highly complex and involves helping pupils understand something that they can never actually physically experience other than through representations. This suggests that investigating the way in which different representations are used to create mathematical meaning is essential.

Duval (2006) suggests that representations fall into different representation registers. Each register adheres to a certain set of rules and is somewhat akin to a language. Each representational register will have certain rules and processes that govern the way in which it is to be used. Duval suggests that becoming fluent in using different representational registers and moving between them is what promotes mathematical understanding and the ability to problem solve. He describes these as the processes of “treatments” and “conversions” (Duval, 2006: 111). *Treatment* of mathematical representations occurs when someone is using the same representation register and manipulating a piece of mathematics with it, whereas *conversion* refers to the manipulation of an aspect of mathematics by moving between different representation registers (see figure 4 for examples). Duval suggests that the combination of these two things is what leads to someone developing a good understanding of mathematical ideas and being able to solve problems (ibid., 2006). This idea is very close to Zoltan Diene’s notion of multiple embodiments, which suggests that exposing pupils to many different representations will lead them to perceiving the mathematical object in a more complete way (Dienes, 1967).

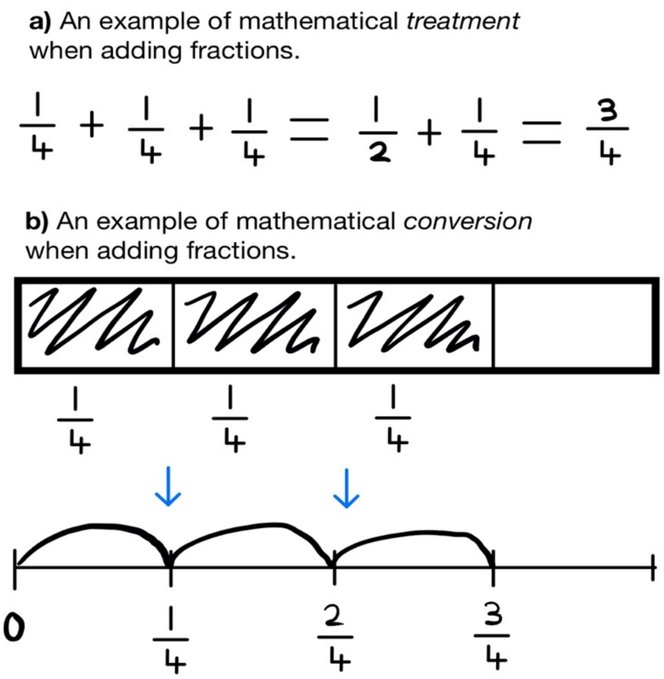
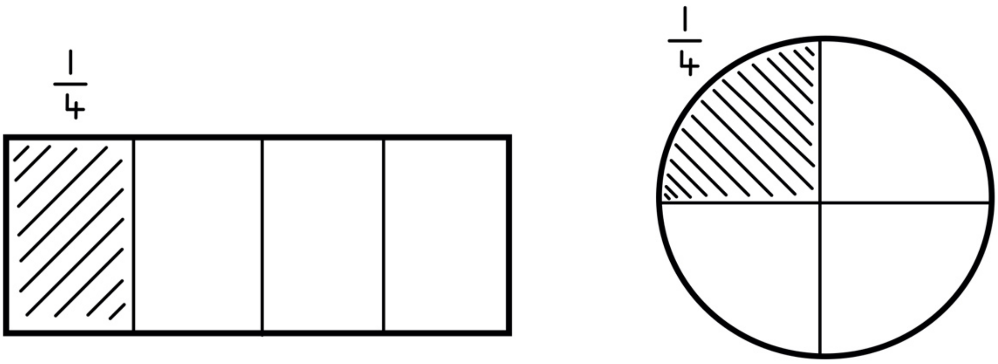


Figure 4 - Examples of 'treatment' and 'conversion' with fractions representations

However, it is difficult to determine how these registers should be differentiated from one another. For example, one study that used the work of Duval as a theoretical basis applied the idea of representational registers to analyse teachers' ability to notice changes between registers in a fictional lesson scenario (Dreher and Kuntze, 2015). In one example given, the teacher is conducting a lesson on fractions and moves from a rectangular representation of one quarter to a circular representation (figure 5) during an exchange with a pupil (ibid., 2015). In the study it is argued that this signifies a change in representational register, and it is suggested that the teacher in question should have made this change more overt to the pupil in order to develop the pupil's ability to convert between different registers. This is a contentious example because, on one hand, the two are clearly different – a circle is different from a rectangle therefore the two representations may be perceived as different by the pupil. However, on the other hand, we might see both of these representations as belonging to the same register because they are both representing the fraction using the area of a 2D shape. This demonstrates that, although the notion of representation registers is a useful one, it is difficult to apply to real-life classroom situations because so much of it depends on individual perception.



A rectangular diagram showing one quarter.

A circular diagram showing one quarter.

Figure 5 - Different Area Model Representations of a Fraction

At this point it is useful to consider what is arguably one of the most widespread frameworks for thinking of different representation registers that is used by teachers in the UK: the ‘Concrete-Pictorial-Abstract’ (CPA) approach (Merttens, 2012). Although not explicitly described as ‘representational registers’, the CPA approach is essentially a way of differentiating between different forms of mathematical representations that may be considered registers. Based on Jerome Bruner’s (1966) theory of enactive (concrete), iconic (pictorial) and symbolic (abstract) representations, teachers are often encouraged to use multiple representations that fall into each different register during mathematics lessons (figure 6 demonstrates some common examples of this often shown to teachers), and in general terms it is encouraged that teachers try and utilise all three types of representation in lessons (Drury, 2018). However, the issue relating to perception also applies here; the CPA framework is somewhat ambiguous and can lead to misleading guidance for teachers (Merrtens, 2012). Should teachers take the idea of concrete representations purely in a real-life sense and only use objects that are real, everyday items? Or should purely mathematical equipment that is physical in nature but not ‘real-life’ per say, also be included? What about photographs of real-life objects? For teachers, these questions are only important if the answer to them affects a pupil’s mathematical understanding. Such questions have been addressed by research where what is described as the ‘perceptual richness’ of objects is investigated. A perceptually rich object is a physical thing that stands out from its environment and invites further exploration of the object itself (Petersen and McNeil, 2013). Using this terminology, a single colour counter would not be perceptually rich whereas a real apple might be. In fact, often, it is the case that specific mathematical equipment is not perceptually rich. The consensus from research evidence presented by some has been that perceptually rich objects are not useful for learning about mathematical ideas because the richness of the object detracts from the mathematical idea it is being used to represent (Carbonneau, Marley and Selig, 2013). However, research has shown that, with young children (around 4 years old), perceptually rich objects can actually facilitate mathematical learning when the objects themselves are unfamiliar to the pupils (Petersen and McNeil, 2013). The combination of these issues illustrates the difficulty teachers may have with using and defining representational registers, even when put in simplistic terms such as 'CPA'. This also poses problems for researchers wishing to adopt the notion of representational registers as a theoretical framework from which to work from.

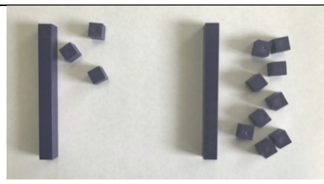
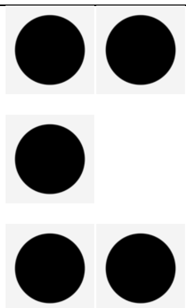
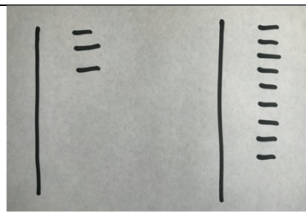
Concrete / Enactive		
Pictorial / Iconic		
Abstract / Symbolic	**5 / five**	**13 + 19 = 32**

Figure 6 - Examples of 'Concrete', 'Pictorial' and 'Abstract' Representations

Despite this ambiguity within Duval's (2006) idea of representation registers, it appears that the theory itself serves an important purpose. That is, to promote thinking amongst educators about the complex and nuanced way in which the use of multiple representations impacts the process of communicating mathematical meaning. In particular, a large meta-analysis adopting the register of concrete representations as its focus (specifically, physical mathematical equipment - ‘manipulatives’) demonstrated a small to moderate effect size in favour of teaching that uses such equipment (Carbonneau, Marley and Selig, 2013). This was particularly evident when testing for retention of mathematical knowledge but less so when testing for ability to solve problems. This analysis backs up Duval’s (2006) theory by suggesting that use of such equipment is likely to have the most impact when pupils are taught specifically how and when to use manipulatives, supporting particularly the idea of *treatment*. However, because the analysis did not consider studies that focused on anything but manipulatives, it can only inform thinking about a very narrow aspect of mathematical representation and still leaves many questions about the efficacy of using combinations of registers. For example, does the use of physical materials improve pupils’ learning even more than the analysis showed, when it is combined with visual diagrams and a careful use of language? Those taking a social constructivist perspective argue that much greater attention is needed on the minute details of the relationship between teachers and pupils, along with the role of language, when multiple representations are being used (Cobb, Yackel and Wood, 1992). Arguably, the ambiguity within the research evidence as to how representations should be used supports the need to investigate these issues further.

2.3.3 A Social Constructivist Perspective on Representation

Those who have approached issues of representation and the nature of mathematical objects from a social constructivist position have highlighted significant issues with the internal and external view of representation promoted by Goldin (Cobb, Yackel and Wood, 1992; Goldin, 1998). Firstly, they argue that such a view of learning mathematics often leads to the “excessive algorithmatization of mathematics” (ibid., 1992: 14). By this they mean that, where there is an over emphasis on pre-existing external representations of mathematical ideas, teachers are more likely to use them in a procedural and algorithmic way, teaching pupils to use them as an aid for mathematical procedures but not to understand the structures they are designed to reveal. Secondly, it is argued that the development of mathematical meaning is both an independent cognitive process as well as a social discursive one - mathematical objects gain collective understanding and meaning through social discourse and are therefore culturally situated (Cobb, Yackel and Wood, 1992; Cobb, 2000; Radford, 2006). The notion of distinct external and internal representations assumes that what a person does not know already exists as a pre-formed external representation ready to be acquired, thus neglecting the social discourse element of mathematics. They highlight the fact that, when external representations are used, there is often the assumption that there is shared understanding of what the representation means and what mathematical object it refers to (ibid., 1992). However, those who already have a shared understanding of mathematics devise most mathematical representations that are used in schools, and if a novice pupil experiences such a representation who is to say they will perceive the same mathematical structure it was designed to show? It would be possible to have a pupil who can use all of the representations in a way that appears effective, with little or no understanding of the mathematical objects they are meant to refer to. Equally it would be possible to have a pupil who has sound understanding of the mathematical objects but does not understand how to use a commonly understood representation system (ibid., 1992). Both scenarios are somewhat analogous to Skemp's (1976) psychological framework for understanding mathematics that highlights two types of understanding: understanding the rules and 'tools' of mathematics without meaning (instrumental understanding) and understanding mathematical relationships (relational understanding). Both objections to the internal and external view of representations suggest that focusing on mathematical discourse is central to understanding the role of representations in the process of mathematical meaning making. Therefore, from this perspective, it is not just the multiple representations themselves that are important to the learning process, but the way in which they are used by learners. This has led to some suggesting that they key focus for research should not necessarily be representations themselves but "representational activities" (Sfard and Thompson, 1994: 2) or "modelling activities" (Van den Heuval-Panhuizen, 2003: 29). In this way the term 'representation' in mathematics might be considered as a verb as well as a noun, because it can refer to an act, as well as a thing in itself (Sfard and Thompson, 1994).

Investigating the issues surrounding mathematical discourse further, Sfard (2000) considers two types of discourse: ‘actual reality’ (AR) discourse and ‘virtual reality’ (VR) discourse. The majority of discourse between humans can be considered as AR. Most of the time, when communicating about something it is possible to physically access that very thing being discussed (Duval, 2006). For example, when studying different parts of a plant in school, the teacher can get a real plant from outside and the class can see in real life each part of the plant. This is what defines AR discourse: the thing being communicated about is a real and accessible thing. In contrast, VR discourse is where the thing being communicated about is not possible to physically access – it is ‘virtual’. However, within VR discourse there are many language similarities. People will talk about something as if it is really accessible, using many of the same types of words and phrases, even though the thing they are discussing is a virtual thing. Mathematics is an example of this – representations of mathematical objects are often used and talked about as if they are the real mathematical objects themselves even though they are not. This reflects Duval’s (2006) dilemma, which is worth re-iterating – mathematical objects can only be accessed through representations, yet none of these representations *are* the mathematical object itself.

In addition to this, many would argue that it is only when a mathematical object has been socially accepted that it becomes part of the domain of mathematics (Hersh, 1999) and in order for this to happen, representations of the object must have been used to facilitate shared understanding. This duality between representation and mathematical objects is what defines VR discourse and suggests that mathematical meaning is created by a complex interplay between the use of representations and the objects they are representing (Sfard, 2000). This poses a clear issue for teachers: if discourse is about something that is not directly accessible to anybody (VR) and only representations can be used, how can anyone be sure that representations are being used by two or more people to refer to the same thing?

Bearing in mind this dilemma, the work of Luciano Meira (1995) investigates the importance of pupils creating their own mathematical representations during the process of problem solving. Somewhat aligned with the view of Sfard (2000), Meira emphasizes the duality that exists between mathematical representation and the cognitive process of mathematical meaning making, suggesting that the process of creating and using representations supports and enhances the understanding of mathematical objects. Meira's (1995: 310) study highlighted that:

> a display [representation] designed on paper has the important function of shaping its designer's activity at the same time that the designer shapes the display itself…

This suggests that pupils designing and using their own representations serves an important role in the process of learning mathematics and perhaps should be emphasized within instruction. There is some evidence that schools are beginning to develop this through the use of pupil journals (Boyd and Ash, 2018b), however it is arguable that the majority of mathematics education in the current educational climate within England still does not give priority to pupils creating and using representations in this way. The limited research seems to suggest that one way of getting around the dilemma, presented both by Sfard (2000) and Duval (2006) of communicating about mathematical objects, is through striking a balance between use of pre-determined mathematical representations, those created by pupils themselves and the careful use of dialogue in the classroom.

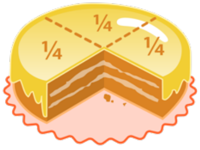
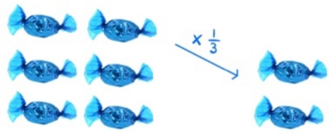
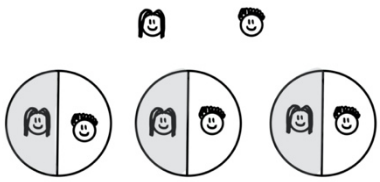
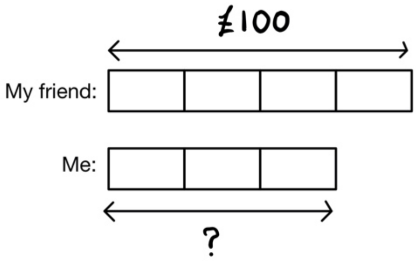
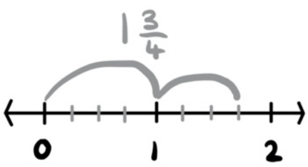
Based on the existing research evidence and the range of theories about developing mathematical meaning, it seems clear that using multiple representations (from different representation registers) combined with pupils' own representations and insightful discourse about these, is likely to be the best possible way of helping pupils deeply understand mathematical objects. The following section will build on this, outlining some of the pertinent issues about representation specifically relating to the domain of fractions.

2.3.4 Representation in the Domain of Fractions

Although the issues surrounding use of multiple representations can apply to many different areas of mathematics, it can be seen that most empirical studies have tended to focus on specific mathematical topics including algebraic functions (Meira, 1995), multiplicative structures (Barmby et al., 2013), geometry (David and Tomaz, 2012) and fractions (Dreher and Kuntze, 2015; Panaoura et al., 2009; Tunç-Pekkan, 2015). This is most likely to be the case because within each mathematical topic there are generally common representations used by teachers and therefore there are likely to be situated belief and knowledge systems related to these that merit study in their own right. This study focuses upon the area of fractions for three main reasons. First, there is already a significant body of research into fractions along with their associated representations (Charalambous and Pitta-Pantazi, 2007; Dreher and Kuntze, 2015; Gabriel et al., 2013; Hackenberg, 2013; Panaoura et al., 2009; Tunç-Pekkan, 2015; Rau and Matthews, 2017) and by continuing this line of inquiry it is possible to contribute to this growing body of knowledge. Second, it is well documented that pupils struggle to learn fractions (Gabriel et al., 2013; Hackenberg, 2013; Siegler et al., 2010) and that teachers often have weak pedagogical subject knowledge of them, particularly when it comes to creating representations (Askew et al., 1997; Ma, 1999). Third, the topic of fractions is an aspect of mathematics that is usually taught in every major phase of education within England (from 6 year-olds to 16 year-olds) and it has been speculated that as a topic, it is an essential basis for later success in school mathematics, particularly in algebraic thinking (Siegler, Thompson and Schneider, 2011; Booth and Newton, 2012). Therefore, the output of this study is likely to have some relevance for teachers in both primary (4 to 11 year olds) and secondary (12 to 16 year olds) schools. One reason as to why the subject of fractions has received attention within the literature is because it is a highly complex area of mathematics that is multi-faceted in nature. This means that attention must be given to the different aspects of the concept and how this relates to representations if it is to be studied in enough detail to be useful for teachers. This section will outline the concept of a fraction according to the literature, consider how this relates to representation and draw attention to some of the empirical research findings that are relevant.

It can be seen that there are different sub constructs that, together, form the concept of a fraction (Charalambous and Pitta-Pantazi, 2007; Hackenberg, 2013; Kieren, 1976). Initially recorded by Kieren (1976), a common approach to identifying these sub-constructs is by analysing them from an expert position and identifying the different constructs that can be found. Although there are slight variations within the literature, these different sub constructs are commonly agreed to be part/whole, quotient, operator, ratio and measure (Charalambous and Pitta-Pantazi, 2007; Charalambolous et al., 2010; Kieren, 1976). Because each sub-construct relates to the different applications of fractions within mathematics, it can be seen that each one is likely to have different representations that are commonly associated with them. Table 4 illustrates these five different sub-constructs, providing examples of how they might be referred to in practice with some possible representations that could be used. It is not the case that each sub-construct has its own discrete set of representations, and there is some overlap (e.g., an area model is used in the example both for part-whole and quotient), however it is likely that some representations are better for exposing certain sub-constructs than others. For example, fractions on a number-line seems like a helpful way to show pupils the concept of fractions as a linear measurement from zero. In addition to these sub-constructs, an added layer of complexity is the linking of arithmetic operations with fractions. Not only are there different sub-constructs of fractions, but we can also compare, add, subtract, multiply and divide them. For example, the division of a fraction by another fraction (e.g., $\frac{1}{4} \div \frac{1}{2}$) is something that even highly qualified maths specialist teachers have been known to struggle with when it comes to creating a real-life scenario that exposes the mathematical structure (Ma, 1999). Therefore, alongside the different sub constructs and representations that could be used, teachers also need to consider which representations are most likely to support pupils conducting such operations on fractions.

Table 4 - The fraction sub-constructs and examples

Sub-construct	Description	Examples
Part-whole	The fraction refers to a specified number of equal parts of a whole.	*"The cake has been cut into four equal parts and I have one part."* 
Operator	The fraction refers to an operation upon another number, causing quantities to change.	*"I had 6 sweets and I ate one third of them."* $\frac{1}{3}$ x 6 
Quotient	The fraction refers to the division of one whole number by another.	*"We shared three pizzas between the two of us."* $3 \div 2 = \frac{3}{2} = 1\frac{1}{2}$ 
Ratio	The fraction refers to the relationship between two quantities.	*"I have $\frac{3}{4}$ as much money as my friend."* 
Measurement	The fraction refers to a specified linear distance from zero in order to measure something.	*"A piece of wood is $1\frac{3}{4}$ metres long."* 

Analysing the sub constructs in this way is useful for some areas of research, such as textbook task analysis, however there is inherent danger in accepting this as the only way because it does not take into account the perspective of pupils as novice learners. To tackle this, it is important to identify a scheme that demonstrates from a pupil's point of view, how fractions are learned. In attempting to do this, others have identified what has been termed a 'fractional scheme theory' that breaks down understanding of fractions into four progressive stages (table 5) (Hackenberg, 2013; Tunç-Pekkan, 2015). Again, as with the sub constructs previously identified, this scheme has an important place as a research tool as well as a tool for teachers when considering the best way to introduce learning about fractions to pupils. Nevertheless, it seems as if fractional scheme theory does not take into account the full breadth of fractions as a concept and focusses primarily on different levels of understanding fractions as parts of wholes. It is hard to see how it relates to the learning of the other fraction sub-constructs, for example seeing a fraction as a measurement from zero.

Table 5 - A summary of the five stages of fractional scheme theory (Tunç-Pekkan, 2015: 422-423)

Phase of fractional understanding	***Description***
1. Parts-within-wholes fraction scheme	Being able to partition a given shape and identify its parts with a fraction symbol (parts may not be equal if drawn). This also includes the ability to relate a pre-shaded fraction of a shape with the correct symbolic representation. The basis of this stage is still counting.
2. Part-whole fraction scheme	In addition to the first phase, this includes what Tunç-Pekkan (2015: 423) refers to as "partitioning and disembedding". This means a pupil can mentally take a part out of the whole whilst still being able to see how it relates to the whole itself. This includes being able to split a given shape up into equal parts, however checking does not occur because there is no iteration of the parts.
3. Partitive unit fraction schemes	This includes phases one and two with the additional element of iterating taking place. This means that a pupil can move fluidly between seeing parts of wholes but also what the whole is from a part by iterating it according to the fraction symbol given. This is with simple unit fractions.
4. Partitive fractional scheme	Referred to as 'genuine' understanding of fractions, this includes all of the first three phases in a more complete and generalised form. Therefore, in this phase pupils can use what they know about part-whole relationships and iterating to find non-unit fractions. For example, to find $\frac{3}{4}$ of a quantity a pupil could find $\frac{1}{4}$ and then iterate it three times.
5. Iterative fractional	According to the scheme, this is the highest level of understanding. It involves the simultaneous use of partitioning and

scheme	iteration. For example, finding $\frac{8}{7}$ of a given quantity by partitioning it into $\frac{1}{7}$ and then iterating this 8 times. This involves the pupil working with three fractions mentally within the same process - $\frac{1}{7}$ (the unit), $\frac{7}{7}$ (the whole) and $\frac{8}{7}$ (the final amount).

Considering these multiple constructs that together form the topic area of fractions, it is not surprising that they are considered to be such a difficult area of mathematics. It is commonly argued that using multiple representations when teaching pupils about fractions is essential because of this (Dreher and Kuntze, 2015; Prediger, 2011), however here we are faced with a dilemma – multiple representations are essential to gain deep understanding yet, there is the potential that they could make a complicated topic even more difficult to learn for pupils because they reveal their multifaceted nature. Nevertheless, studies have highlighted that multiple representations can be used in specific ways to support the learning of fractions. Three studies merit particular attention because of their focus on use of representation and its impact on learners when learning about fractions.

Zelha Tunç-Pekkan (2015) utilised the fractional scheme theory described in table 5 in order to identify particular graphical representations that would support pupils' learning at each phase. She designed problems using either circular, rectangular or number-line representations of fractions and analysed pupil responses. The findings suggested that at each phase, some representations were more appropriate than others and that, in general, pupils struggled to apply fractional knowledge using a number-line representation when compared to circular or rectangular models (ibid., 2015). This suggests that perhaps a number line representation is more useful once pupils have become comfortable with circular and rectangular representations and raises questions about the order in which common representations should be used and how quickly new ones should be introduced.

In another study, Rau and colleagues (2009: 442) speculated that prompting pupils to "self-explain" whilst using multiple visual representations to learn fractions would lighten some of the cognitive load added by their use. The theory was that prompting pupils to spend more time thinking about and understanding how particular representations related to the concept would mean that their use would not add to the cognitive demand of working through fraction problems. They found that in prompting pupils to do this, multiple representations had a greater impact on performance measures. Here it is important to highlight the fact that 'self-explanations' are taken to mean explicit opportunities where pupils are prompted to provide explanations about how they solved a problem and what the role of the different representations were (ibid., 2009). This appears to be quite similar to the practice of journaling in mathematics being adopted by teachers within schools in England adopting a mastery approach (Boyd and Ash, 2018). It also seems to support some aspects of the social-constructivist arguments relating to representation that highlight the importance of carefully thought-out dialogue supporting the use of representations, or 'representational activities' (Cobb, Yackel and Wood, 1992; Sfard and Thompson, 1994; Sfard, 2000). Nevertheless, Rau and colleagues (Rau et al., 2017; Rau and Matthews, 2017) have gone on to further investigate the effectiveness of multiple visual representations in learning fractions and have summarised what they describe as “representational competencies” that they refer to as “visual understanding”, “visual fluency”, “connectional understanding” and “connectional fluency” (Rau and Matthews, 2017: 535). These appear to align to Duval's theory of representational registers. Specifically, visual fluency and understanding relate to Duval's idea of 'treatment' in that they refer to a pupil's ability to read and understand a particular representational format. Whereas connectional understanding and fluency relate to Duval's idea of 'conversion' in that they refer to a pupil's ability to translate between multiple representations (Duval, 2006: 111). Again, this supports the use of multiple representations alongside teacher and pupil dialogue to aid the effective communication of mathematical meaning.

Further research into the use of representations when teaching fractions has been done as part of the Rational Number Project (RNP) in the USA (Cramer, Post and delMas, 2002). The RNP was designed to help pupils develop better understanding of rational numbers, of which fractions is a major part, primarily by getting teachers to use a wide range of representations in their lessons. Based upon a model derived from the work of Bruner (1966) and Dienes (1967), the RNP project focuses on teaching pupils how to make translations between different representations so that they develop a deep understanding of the concept and are able to solve problems more effectively (Cramer, Post and delMas, 2002). Interestingly, although not based upon his work, this seems to be aligned with Duval's (2006) theory of representational registers discussed earlier because the suggestion is that pupils need to be able to move between different types of representation registers to develop deep understanding of concepts. One of the most pertinent studies compared the impact of teaching within the RNP to pupils who had received standard, textbook based, teaching in the same district (Cramer, Post and delMas, 2002). They found that pupils who had been taught using the RNP techniques outperformed those who had not when it came to fractions related questions (ibid., 2002). Specifically, two things are of note. First, pupils in the group who received RNP instruction outperformed the control group on questions requiring application of operations, even though they had received less time learning these in class compared to the control group. Second, within the RNP materials, pupils were taught to spend a significant amount of time verbally reasoning about different representations and how they relate to one another, which seems very similar to what Rau and colleagues (2009) describe as 'self-explaining' and again supports arguments about the importance of dialogue when using representations (Cobb, Yackel and Wood, 1992; Sfard and Thompson, 1994; Sfard, 2000).

In summary, the literature about teaching fractions and using associated representations supports the broader research by suggesting that using multiple representations is important, however there are complexities about how to use them most effectively. Of note, it appears that pupils need to be taught in such a way to be able to translate from one type of representation to another and it is the dialogue that accompanies this type of activity that is essential to its success. Within this study, I take the stance that high quality teacher knowledge of fractions involves understanding each of these sub constructs and being able to work with them, using a broad variety of representations. This does not necessarily mean being able to name the sub constructs per say, rather, being able to identify them and notice the differences between them and how they all relate to the abstract idea of a fraction. Being able to identify different representations and how these might link with one another is key to this. Nevertheless, considering the findings emanating from research associated with the RNP (Cramer, Post and delMas, 2002), it seems likely that the curriculum materials used by teachers is likely to impact upon what and how representations are used. Therefore, it is important to consider the role of textbooks in teaching.

2.4 Textbooks and Teaching for Mastery

An important factor in my study is the introduction of government recommended textbooks for primary schools (pupils aged 4-11 years) in England (NCETM, 2019). This forms part of the national mathematics education landscape in England where what is termed 'teaching for mastery' is being developed across state funded schools. Currently, over 50% of English primary schools have engaged in government funded professional development programmes designed to promote this approach (NCETM, 2023a). In this section, because of its contextual significance, I will first outline the idea of teaching for mastery in relation to the English education system and highlight how it forms part of the contextual backdrop to my study. This will then lead on to a critical discussion of the role of textbooks in school maths, which plays a central role within my study.

2.4.1 Teaching for Mastery

'Mastery' as an educational approach is contested, however there is some evidence of positive impact about teaching referred to as 'mastery', where a whole group or class must achieve an acceptable level of competence (decided by the teacher) before anyone moves on from the topic being studied (EEF, 2023). Central to this approach to teaching appears to be a belief that, with appropriate conditions for learning, almost all pupils are capable of learning the curriculum content, which is an idea that has been explored within educational literature since the 1970s (Block & Anderson, 1975; Bloom, 1982). In relation to school maths in England, ‘teaching for mastery’ is more distinct and defined than the aforementioned ‘mastery’ approaches and can be seen as a curriculum reform movement that is a central part of the context of my study. Although it is distinct from the general idea of ‘mastery’, it does share the same belief as its basis, which is that all pupils are capable of learning school maths and that teachers need to find the conditions for learning that will lead to this desired outcome (NCETM, 2023b). In essence, the concept of ‘teaching for mastery’ attests that school maths should lead to all pupils being able to learn the curriculum content and that suitable curriculum enactment on the part of teachers should be developed to enable this to happen. Nationally, in England, ‘teaching for mastery’ appears to be influenced by a mixture of research informed practice, alongside influence from high attaining Southeast Asian regions such as Shanghai and Singapore (Boyd and Ash, 2018a, 2018b). For example, the promotion of using multiple representations to teach school maths is a core aspect of ‘teaching for mastery’ (Gear, 2022; NCETM, 2023b) and has a rich evidence base that has been discussed previously (section 2.3). However, some aspects of the approach within England, such as the Mathematics Teacher Exchange, which was an exchange programme between teachers from England and Shanghai (Boylan et al., 2019), seem to be based upon looking to the success of other nations rather than empirical research evidence of teaching approaches. Some dispute the effectiveness of ‘teaching for mastery’ and highlight that there has been little research into it as a whole approach within England (EEF, 2023). Despite this, some early research identified that moderate gains could be made to pupil learning when a teaching for mastery-based scheme was used (Jerrim and Vignoles, 2016). Alongside this, I have undertaken research, before this study, where it was identified that teacher beliefs and practices had shifted positively because of adopting a ‘teaching for mastery’ approach with a textbook scheme alongside it (Boyd and Ash, 2018a, 2018b). Arguably, because ‘teaching for mastery’ has a broad definition which includes a set of beliefs alongside a wide-ranging set of practices (NCETM, 2023b), it is more pertinent to study some of the practices individually rather than attempting to study it as one unified approach to teaching. Within my study, I aim to study one specific aspect that is included within the approach, which is the use of multiple representations. Additionally, although textbook use is not cited as a necessary factor in adopting a 'teaching for mastery' approach in the national definition (NCETM, 2023b), the government decision to formally approve and offer match-funding for schools to buy them does suggest their important role in the 'teaching for mastery' curriculum reform movement within England.

2.4.2 Textbooks and School Maths

As previously highlighted, the use of government approved textbooks within England is a central aspect of current curriculum reforms that are related to the idea of 'teaching for mastery'. This is of particular importance because of the complex role that textbooks play in the process of turning the subject of mathematics into school maths. This process is what Bernstein (2000) refers to as knowledge recontextualisation, which involves taking a knowledge domain (such as mathematics) then selecting and sequencing aspects of it so that it becomes a school subject. This is a process that inherently involves value laden choices as to what knowledge should, or should not be, included in a school curriculum (ibid., 2000). As Lilliedahl (2015: 41) points out "choosing content involves selecting an offer of meaning". The textbook itself has been written by authors who will have had to make many choices of this nature such as what content to include or not, how much of each thing to include and what type of representations to use. This means that, in schools where a textbook is being used, it is playing an important part in the process of recontextualisation and therefore must be considered within the research design of this study.

Many highlight that textbooks have a significant role to play in determining the potential opportunities for pupils to engage with different aspects of mathematics, such as representations (Charalambous et al., 2010; Wijaya, Heuvel-Panhuizen and Doorman, 2015). However, in trying to pinpoint the specific role of the textbook it is necessary to consider other perspectives of the school maths curriculum and how these all fit together. Arguably, textbooks form an interpretation of the formal written curriculum, and this seems to be the case with the textbooks that the English government is currently recommending, which are said to cover the statutory curriculum content (NCETM, 2017). When being used by teachers in the process of teaching, this interpretation of the statutory curriculum (the textbook) must then be interpreted again, by a teacher, in the process of planning and delivering lessons. Within the literature these different perspectives on curriculum are mirrored in the terms “planned curriculum” which refers to goals, content and activities outlined in policy, the “curriculum as enacted by teachers” which refers to how the teacher plans and delivers the curriculum and also the “experienced” curriculum which refers to what the pupils actually experience as the curriculum (Gehrke, Knapp and Sirotnik, 1992: 55). Because this study is focussed upon the beliefs, knowledge and practice of teachers, it is the first two of these that are of particular interest. Applying Bernstein’s (2000: 33) theory of knowledge recontextualisation, the so-called formal curriculum might be seen as an “official recontextualising field” (ORF) that takes the discourse of mathematics and uses official sources to change it into school mathematics contained within the National Curriculum. This official recontextualising field exists alongside what Bernstein (2000: 33) calls the “pedagogic recontextualising field” (PRF) which would include the intended and enacted curriculum. Essentially, the pedagogic field is where knowledge is recontextualised by those working in schools through the work of being a teacher (ibid., 2000). According to Bernstein, the PRF can act independently of the ORF as well as being influenced by it, therefore the relationship is both one of autonomy whilst also being a power struggle. What is important here is to consider where the textbook fits in – is it part of the ORF or the PRF? On one hand, we might consider it as part of the ORF because it is approved by a government organisation and written by a collection of experts in mathematics teaching. However, it is also perhaps an example of a pedagogic device as it is used by teachers on a daily basis to help shape what happens in lessons and is therefore an important part of the PRF. In other words, the intentions of the textbook authors are likely play an active role in the teacher’s enactment of the curriculum, thus creating a fuzzy line between the ORF and PRF. For example, a textbook, which is written by authors whose expertise in mathematics is beyond that of an average primary teacher, may well present representations of mathematical objects in such a way that is unfamiliar to teachers and therefore prompt the development of knowledge and beliefs, thus influencing classroom practice. This is an important point to highlight in relation to this study – the research design will need to enable the collection of data about how the textbook is influencing the teacher. Nevertheless historically, teachers in England have been estimated to only use a textbook as the main basis for mathematics planning 10% of the time, with the other 90% involving substitution with other curriculum materials (Mullis et al., 2012). This suggests that the extent to which teachers in England, who are using a textbook, are enacting the intended curriculum of the textbook authors is questionable. Despite this, recent research in England has shown that some teachers are using the textbook with little or no substitution for other materials and that this is having an impact on their beliefs and classroom practices (Boyd and Ash, 2018).

This makes it imperative that one aspect of my study focuses on the complex interplay between the textbook and the enacted curriculum asking the question ‘how do individual teachers interact with, use and are influenced by curriculum materials such as textbooks?’ This is what Remillard (2005: 212) describes as “curriculum use”, pointing out that this rests upon the assumption that there is interaction of some sort between teacher and materials. Nevertheless, the study of curriculum use of a textbook is complex and there does not appear to be a substantial theoretical basis on which to build upon. Sebastian Rezat (2006) argues for an extension of activity theory highlighting that the typical ‘subject-mediating artifact-object’ model for cultural activity lacks sufficient depth to fully explain the role of textbook use in mathematics classrooms. He suggests that using a three-dimensional tetrahedron model that has three triangular faces incorporating the *textbook*, *teacher*, *students* and *mathematical knowledge* is a more accurate way of describing textbook use in mathematics lessons (ibid., 2006). Alternatively, Remillard (2005) proposes a theoretical framework that highlights the teacher as *participating* in curriculum use of a textbook. This refers to the idea that teachers are not simply following or interpreting the textbook, they are participating in the enactment of mathematics teaching along with the textbook (ibid., 2005). Although both models highlight the complexity of textbook use, they also both seem to treat the textbook as a passive material that is used (to varying degrees) by the teacher and pupils. In contrast to this, Hetherington and Wegerif (2018) draw upon the work of Karen Barad and propose that physical materials used in the classroom (such as the textbook) should be considered as mutually constitutive in the creation of meaning along with both teachers and pupils. They suggest that materials (or matter) and people engage in a complex, entangled discourse during the process of meaning making, emphasizing that materials are not given individual agency but are instead part of this process, which they describe as “material-dialogic pedagogy” (ibid., 2018: 27). This approach to classroom materials and their place in the process of meaning making is informed by Karen Barad’s (2007) theory of agential realism.

Barad (2007) uses examples from quantum physics to exemplify the interconnected nature of the social and physical realms. She highlights that the majority of mainstream theories and philosophies about the world consider the two (social and physical) as separate entities, with the social realm often claiming a more significant role. For example, activity theory, which other models of textbook use are based upon (Remillard, 2005; Rezat, 2006), treats physical objects as artifacts (although these can also be abstract in nature) that *mediate* meaning as part of social activity. In contrast to this, within the theory of agential realism the two are entangled and inseparable from one another. Using the terminology of ‘matter’ to refer to all things physical in our world (both man-made and natural), she argues that matter itself plays a significant role in the process of meaning making and discourse: matter is not a mediator of meaning, it is *part* of the meaning and, with different matter, meaning would be different. Within the context of this study, this may refer to everything from the architecture and design of the classroom to the textbook or manipulatives being used in a lesson – all would be considered matter. Essentially, the theory proposes that all these physical things (the ‘matter’) matter more than is often thought, they are an active part of the meaning making process going on in a classroom. She views activity not as being between separate entities (such as humans and materials), described as *inter*-activity, instead she borrows the term “*intra*-activity” from theoretical physics to propose that there are not distinct boundaries between entities. Both matter and meaning are “mutually articulated” and neither one can be said to have come before the other, neither ontologically nor epistemologically (Barad, 2007: 152). Both matter and human beings are affecting each other in some way or another at any one point in time and this is how meaning is created. In short, you cannot have matter without meaning but you also cannot have meaning without matter.

In summary, textbooks matter. They are an important part of the school maths classroom and may potentially have significant influence over the way in which mathematical meaning is communicated. Within my study I aim to better understand the role that the textbook plays in influencing the way in which teachers choose and use representations to communicate mathematical meaning. This will mean that the research design will need to consider the contents of any textbook being used in the form of a textbook analysis, as well as opportunities to gather data about how the textbook is perceived by the teacher.

2.5 Literature Review Summary

This chapter has outlined the wide-ranging areas of literature than are pertinent to this study. In particular, it has highlighted a number of issues related to the contested nature of mathematics, teacher beliefs and knowledge of mathematics, representations, and the role of textbooks that, together, help identify a gap in knowledge that this study aims to contribute to. Altogether, some key conclusions can be drawn from these areas that relate strongly to the recontextualisation of mathematics into ‘school maths’ and, more precisely to the research questions that are the focus of this study.

Approaching the subject of mathematics from a philosophical angle, it can be seen that the very epistemological and ontological nature of the subject is contested (Lakatos, 1976; Ernest, 1991; Hersh, 1999; Charalampous, 2016).

Nevertheless, there seems to be a prevailing stance within the education literature that takes mathematics to be socially constructed, thus fallible, but still an actual body of knowledge that has generalised meaning within the mathematics community; sometimes referred to as 'quasi-empiricism' (Putnam, 1975; Lakatos, 1976; Ernest, 1991). This stance also appears to be in line with more recent, broader social realist arguments about the place of knowledge in education and society (Moore and Young, 2009; Maton, 2014; Lilliedahl, 2015). This is important to this study because such an understanding of mathematics does not seem to be reflected within western culture (Boaler, 2016) and beliefs about the nature of mathematics are varied amongst teachers (Thompson, 1992; Erikson, 1993; Raymond, 1997; Leatham, 2006). This study is concerned with the recontextualisation of mathematical knowledge into school maths and some have argued that a lack of understanding about the nature of mathematics amongst teachers can lead to the school version of mathematics having a "low epistemic quality", meaning that it does not reflect the broader domain of mathematics as it exists in the world outside of school (Hudson, Henderson and Hudson, 2015: 377). Nevertheless, it still appears as though more research is needed in order to better understand the importance of these philosophical arguments to the classroom practices of teachers.

Leading on from this, the literature on teacher beliefs and knowledge highlights a need for further research into this area. Namely, because much of the research has focussed either on beliefs or knowledge and often come to inconsistent conclusions about the relationship between beliefs, knowledge, and teaching practices (Muis, 2004; Philipp, 2007; Hill et al., 2008; Sleep and Eskelson, 2012). This is likely to be due to a number of reasons, but one that is of key importance here is the difficulty with making a distinction between beliefs and knowledge (Fennema and Franke, 1992; Petrou and Goulding, 2011). For this reason, more recent research has tended to take both into account (Sleep and Eskelson, 2012; Kuntze, 2012; Dreher and Kuntze, 2015), however further research is needed to develop understanding in this area. Nevertheless, the literature does highlight a range of beliefs and knowledge (related to fractions in this study) that lead to better teaching and learning (Kloosterman and Cougan, 1994; Hofer, 1999; Muis, 2004; Ball, Thames and Phelps, 2008; Dreher and

Kuntze, 2015; Bonne and Johnston, 2016). In relation to beliefs, Muis (2004) uses the term ‘availing’ to refer to beliefs that research shows to lead to better teaching and learning. In this study, because beliefs and knowledge are being studied together as one system, the term ‘availing’ will be used to refer to the beliefs and knowledge that research has demonstrated to be important. Alongside this, beliefs and knowledge pose a highly complex area of study due to the many different types of beliefs and knowledge that are outlined within the literature (Kieren, 1976; Schulman, 1986 Törner, 2002; De Corte, Op't Eynde and Verschaffel, 2004; Philipp, 2007; Ball, Thames and Phelps, 2008; Kuntze, 2012; Hackenberg, 2013). This study is primarily concerned with teachers' beliefs and knowledge and its relationship to classroom practice. Therefore, aspects from the literature that relate specifically to this, rather than the beliefs of pupils for example, will be of primary concern here. In particular, Kuntze's (2012) model for investigating beliefs and knowledge in tandem will be utilised so that a teachers' belief and knowledge systems can be studied to provide a more multi-faceted approach than previous research. Further explanation of this can be found within the theoretical framework of this study (chapter 3).

In addition to teacher beliefs and knowledge, this study focusses specifically on the use of representation in mathematics. This area within the literature has received significant attention and relates to the ontological nature of mathematics as well as to complex practical concerns for teachers (Cobb, Yackel and Wood, 1992; Goldin, 1998; Duval, 2006; Radford, 2006; Carbonneau, Marley and Selig, 2013; Petersen and McNeil, 2013; Zelha Tunç-Pekkan, 2015). Regardless of philosophical stance, it can be seen that there is widespread agreement that mathematics is a body of knowledge that is comprised of abstract knowledge objects (Hersh, 1999; Sfard, 2000; Duval, 2006; Radford, 2006). This situates mathematics aside from many other subjects because the objects of study are not accessible other than through representations (Sfard, 2000; Duval, 2006; Radford, 2006). This poses a dilemma for teachers – how can pupils understanding be developed when the objects of study can only be accessed through representations? (Duval, 2006). It is for this reason that much of the literature espouses the use of multiple representations as an effective teaching technique and much of the empirical research supports this (Dienes, 1967; Cramer, Post and delMas, 2002; Prediger, 2011; Carbonneau, Marley and Selig, 2013; Dreher and Kuntze, 2015). Nevertheless, this is not without complex caveats about the way in which they are used. Firstly, there is strong evidence to suggest that teachers should always use representations with a mathematical purpose, and this should be made clear to pupils (Cramer, Post and delMas, 2002; Carbonneau, Marley and Selig, 2013). Alongside this, there is evidence that pupils need to develop the ability to translate between different representations and also work within one type of representation system, learning to manipulate these to aid their mathematical thinking (Duval, 2006; Carbonneau, Marley and Selig, 2013; Dreher and Kuntze, 2015; Rau et al., 2017; Rau and Matthews, 2017). There is also tentative evidence, along with strong theoretical argument, suggesting that pupils should be given opportunity to generate their own representations (Sfard and Thompson, 1994; Meira, 1995; Van den Heuval-Panhuizen, 2003). Underpinning all of these recommendations seems to be the importance of dialogue. In much of the theory and research it is suggested that teachers need to think carefully about how representations are used and discussed within the classroom as this seems to underpin the use of representations to facilitate the meaning making process (Cobb, Yackel and Wood, 1992; Sfard, 2000; Cramer, Post and delMas, 2002; Rau et al., 2009).

Finally, it has been shown that the use of textbooks forms an important contextual element to this study. Textbooks themselves operate in an unusual space in that, in some ways, they are an official curriculum document operating in what Bernstein (2000) calls the Official Recontextualising Field; written by mathematics experts to align with the English National Curriculum (DfE, 2013a). Yet, in other ways they are utilised by teachers to create the less official, enacted curriculum, or what Bernstein (2000) refers to as the Pedagogic Recontextualisation Field. This means that they are an important piece of the recontextualisation puzzle and must be considered within the research design of this study. Not only are textbooks important because they are currently quite prominent in mathematics within the English education system, but it has also been highlighted that they are also likely to influence the use of representations and, in this way, the intentions of the textbook authors may play some role in the process of using representations to negotiate the meaning of fractions within the classroom.

The conclusions drawn from this literature review have been developed further so as to form part of this study's theoretical framework (chapter 3). The following chapter will outline this in more detail, explaining how the literature will influence the specific methodology used in this study.

3 Theoretical Framework

The theoretical framework of this study consists of several integrated component parts that all stem from the meta-theory of social realism. Within this section, social realism will first be introduced in order to set the backdrop for the more practical aspects of the framework. Following this, the data instruments derived from the literature review will be presented as well as the introduction of Legitimation Code Theory (LCT) (Maton, 2014) as the final element of this framework.

3.1 Social Realism

Social realism is considered to be a “broad school of thought” rather than a distinct ‘ism’, meaning that within the relevant literature there are some differences between the views held by prominent social realists (Maton and Moore, 2009: 1). It is a sociological theory that draws from critical realist philosophy (Moore, 2013) and argues that knowledge has a real existence, but that the way in which we can access this is fallible (Moore and Young, 2009; Maton, 2014). As humans, our ways of knowing are bound by historical and cultural factors that exert influence over how we understand things (Maton, 2014). Nevertheless, perhaps the most important element of this school of thought is that it attempts to avoid the dichotomy, which it considers to be false, between positivism and constructivism that has pervaded much education research (Maton and Moore, 2009; Moore and young, 2009; Maton, 2014; Lilliedahl, 2015). Social realists argue, despite knowledge being of central importance to education, that it is the study of knowledge itself and its forms and effects, that has long been missing due to an unnecessary focus upon either positivist or constructivist viewpoints (Wheelahan, 2010; Howard and Maton, 2011). On the one hand, much of education research can be seen to take a constructivist stance where the focus is not on knowledge itself, but on the knowers and their relationship to knowledge. Maton (2014: 4) describes this as the "subjectivist doxa" arguing that this view of knowledge as purely subjective, and distilled to a process of knowing, contributes to "knowledge blindness" in research. In contrast, he points out that positivism, the other end of the dichotomy, presents knowledge as value free, decontextualized, and absolute in nature, a sort of currency to be dealt with (ibid., 2014). The problem here is that when education researchers are faced with this choice between positivism and constructivism, they tend to choose the latter thus "dissolving knowledge", which leads to a distinct lack of research into knowledge itself (knowledge blindness) (Maton, 2014: 6). Social realists argue that, instead of this either/or approach, it is better to adopt a "both/and" approach (Maton and Moore, 2009: 2). In practice, this means conducting research on the basis that there is such a thing as knowledge (knowledge exists) but that we access this in a socially bound way, knowledge is a social phenomenon (ibid., 2009).

Nevertheless, it is important to be careful in adopting social realism as a meta-theoretical framework because of its multifaceted nature. Therefore, it is necessary to define the social realist stance towards knowledge that I take in this study. One of the potential problems with some of the literature on social realism is that it attempts to draw a sharp distinction between everyday experiential knowledge and de-contextualised theoretical knowledge (Young, 2013), as if knowledge exists within distinctly separate, rather than inter-connected bodies that relate to everyday social activity. With regards to mathematics and the literature on mathematical objects and representation, it can be seen that this sharp distinction is perhaps unhelpful. For example, how is it possible to draw a line between an informal, contextualised knowledge of the number three and a de-contextualised, theoretical understanding of it? Is it when a pupil can use the concept to relate to many different situations? If so, surely this knowledge is directly contextualised to some extent and always connected to the 'here and now'? Additionally, is there any possible way to define when it has become theoretical rather than experiential knowledge? If the knowledge of the number three needs to relate to *all* possible representations of it to be true theoretical knowledge, then perhaps far fewer of us really 'know' the number three as well as we thought. Therefore, in this study I do not attempt to draw such a sharp distinction and adopt social realism by approaching knowledge as multi-dimensional and something that exists as a social phenomenon bound by historical and cultural factors (Maton, 2014; 2016).

3.2 Connecting Theory and Data

One of the key concerns for social realists is over the connecting of empirical data to underpinning theory (Bernstein, 2000; Maton, 2014), and this appears to be a common thread that runs through many of the relevant studies described in the literature review chapter here. Most studies adopt what Maton and Chen (2016: 30) describe as theoretical “data instruments” that help any theory influence the design of research methods, but these studies then struggle to make the data ‘fit’ with the theoretical framework used. An example of this can be seen in several of the studies about teachers’ mathematical beliefs discussed earlier (e.g., Erikson, 1993; Raymond, 1997) where the data seemed to pose questions that were unable to be answered using the chosen theoretical framework, leading to conclusions about contrasting beliefs that are not much use to teachers. This is one of the reasons why there is a need for research into this area. This relates closely to Bernstein’s (2000: 445) argument that there is often a “discursive gap” between theoretical underpinnings and actual data produced by a study. Maton (2016) suggests that this is more often than not a problem caused by the theoretical frameworks themselves. Applying Bernstein’s (2000) idea that any theory has both an internal and external language to it, Maton (2016) suggests that it is the latter where the problem often lies. Many theories can be seen to have a strong internal language of description, meaning that they are well defined using their constitutive component parts. However, they are often weak when it comes to any “external language of description” (Bernstein, 2000: 132), leading to an ambiguous relationship with the actual data produced by empirical studies, hence the aforementioned discursive gap (Maton, 2016). A strong external language should be able to bridge this gap by its ability to connect up the concepts within the theory with any referents contained within data (Maton and Chen, 2016).

Put simply, it prompts the question - how closely are the underpinning theory of a study and its data connected, and how clear is this? Maton (2016) argues that all too often, education research does not focus enough upon the 'external language' that connects theory and data. This study attempts to overcome this issue by using data instruments derived from the literature, alongside Maton's (2013; 2016) 'Legitimation Code Theory' (LCT) as an explanatory framework.

3.3 Data Instruments

Although Maton and Chen (2016) argue that data instruments alone are not sufficient in providing translation between theory and data, they do however highlight their usefulness. Describing them as providing a "methodological guide to a project by delineating how concepts suggest foci for data collection and questions for analysis", such instruments can be seen as highly useful for designing data collection methods and aspects of data analysis (ibid., 2016: 30). The literature review in the previous chapter has highlighted several pertinent issues for this study that must be taken into account when conducting the research. First, Kuntze's (2012) model of teacher beliefs and knowledge (figure 7) provides a useful tool for designing methods of data collection. Second, the literature about effective use of representations and teacher beliefs and knowledge also provides important implications for data analysis. These will be outlined below and act as key instruments within the process of method design and data analysis. Details of the specific way in which they are used will be outlined within the methodology chapter therefore, here, they are just presented as one aspect of the broader theoretical framework.

Within this study, Kuntze's (2012: 275) framework for teacher beliefs and knowledge will be used to aid the design of data collection methods. As has been highlighted within the literature review, the majority of research into this area has focussed primarily upon either teacher beliefs or teacher knowledge (Shulman, 1986; Schoenfeld, 1989; Kloosterman and Cougan, 1994; Ma, 1999; De Corte, Op't Eynde and Verschaffel, 2004; Hill, Schilling and Ball, 2004;

Kuntze, 2012; Sun, 2015), yet there is evidence to suggest that separating the two is arguably not helpful as each one influences the other (Hill et al., 2008; Sleep and Eskelson, 2012). Therefore, Kuntze's (2012) model forms part of the theoretical framework of this study in that it acts as a tool to help guide aspects of the methodology design. Specifically, this study will collect data both about knowledge and beliefs, as well as at different levels of globality, as is suggested by Kuntze's model (2012).

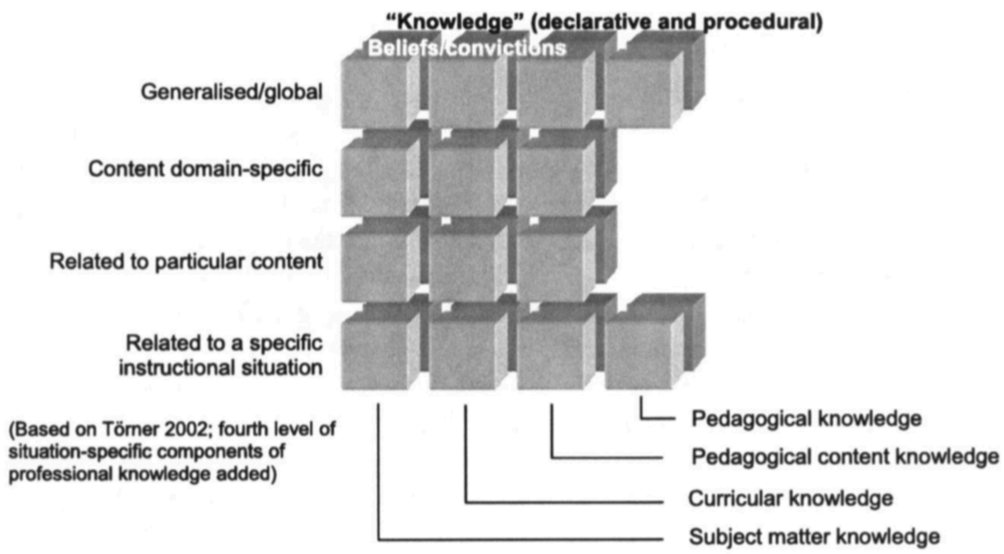


Figure 7 - Data instrument part one - Kuntze's (2012) theoretical model of teachers' professional knowledge

Alongside the use of Kuntze's (2012) model, this theoretical framework makes use of other findings from the literature in the form of theoretical instruments for data analysis. Below, the literature review findings are synthesized into two key areas (effective use of representations, and availing beliefs and knowledge) that will form part of the approach to data analysis within this study, thus acting as data instruments.

The literature review chapter has highlighted that the use of different representations to teach mathematics is potentially beneficial to pupils, nevertheless there are stipulations about the way in which they should be used (Duval, 2006; Carbonneau, Marley and Selig, 2013). Specifically, there seem to be several things that might be considered as effective use of representations. Here, the term 'effective use' is taken to mean representations being used in such a way that they help pupils develop a deep understanding of the mathematics and consequently solve related problems and make connections to other areas of maths. In consultation with the literature, there are a number of key issues related to how representations are used that constitute such effective use and these are summarised along with the supporting literature in table 6.

Table 6 - Data instrument part two - A summary of the effective use of representations

Effective Use of Representations	Supporting Literature
Teachers use *multiple* representations and help pupils make connections between them.	Bruner, 1966; Dienes, 1967; Cramer, Post and delMas, 2002; Duval, 2006; Rau and Matthews, 2017
Multiple representations are used for the purpose of helping pupils develop a deep understanding of the abstract concepts of mathematics.	Cobb, Yackel and Wood, 1992; Sfard, 2000; Duval, 2006
Representations are treated as discussion points in their own right and the reasons for using them are made explicit	Cobb, Yackel and Wood, 1992; Cobb, 2000; Radford, 2006; Carbonneau, Marley and Selig, 2013; Rau and Matthews, 2017
Teachers are explicit when making translations between different representations	Duval, 2006; Dreher and Kuntze, 2015; Rau and Matthews, 2017
Teachers treat representation as a broad concept, acknowledging pupils' own perceptions and affect as part of this	Goldin, 1998, 2002b
Teachers allow time and actively prompt pupils to verbally reason about how they are thinking about representations	Cobb, Yackel and Wood, 1992; Pape and Tchoshanov, 2001; Rau et al., 2009; Rau and Matthews, 2017
Teachers allow opportunities for pupils to develop their own representations	Meira, 1995
Representations are used with a clear mathematical purpose	Carbonneau, Marley and Selig, 2013

Along with the above synthesis of the research into using representations in mathematics teaching, the literature review also highlighted that certain beliefs and knowledge are important for teachers. Muis (2004) adopts the term 'availing' to describe beliefs that the research has shown to positively impact mathematics teaching and learning. In this study, I will adopt the same terminology but broaden it to also include the sort of knowledge that the literature suggests is important for teachers to have. This is primarily because of the blurred lines between what is a belief and what is knowledge (Fennema and Franke, 1992; Kuntze, 2012). Below (table 7) is a synthesis of the literature about availing beliefs and knowledge with specific reference to fractions as this is the focus area of this study.

Table 7 - Data instrument part three - A summary of availing knowledge and beliefs about teaching mathematics

Availing Beliefs and Knowledge	Supporting Literature
Fallibilist beliefs about the nature of mathematics (that it is a subject created and developed through social interaction and not absolute)	Schommer, 1990; Cobb, Yackel and Wood, 1992; Erikson, 1993; Muis, 2004;
A belief in teaching mathematics in a way that helps pupils understand its interconnected nature and develop a broad understanding of what it means to do mathematics.	Erickson, 1993; Muis, 2004; Sun, 2015; Boaler, 2016
Knowledge of the five fraction sub-constructs: part/part whole, quotient, operator, ratio and measure	Kieren, 1976; Charalambous and Pitta-Pantazi, 2007; Hackenberg, 2013
Knowledge of multiple representations that can be used to help pupils understand the concept of a fraction	Fennema and Franke, 1992; Charalambous and Pitta-Pantazi, 2007; Cramer, Post and delMas, 2002; Dreher and Kuntze, 2015; Gabriel et al., 2013; Panaoura et al., 2009; Tunç-Pekkan, 2015
Knowledge of how pupils typically gain an understanding of fractions – including common errors and the type of representations that can support this.	Fennema and Franke, 1992; Ball, Thames and Phelps, 2008; Hill, Ball and Schilling, 2008; Tunç-Pekkan, 2015

Although these data instruments are of significant use within this study because they help guide the focus for method design and data analysis, they do fall short in that, used in isolation, they do not allow any contribution to theory beyond the direct implications for teachers teaching fractions. Therefore, an additional component was sought that would offer greater explanatory power. After considerable reading around a range of different theoretical frameworks, I encountered Legitimation Code Theory (LCT) and it resonated with different aspects of this study. The subsequent sections will first outline the rationale for why LCT has been adopted and then provide an outline of what LCT is, and the specific aspects of it that form the final component of this theoretical framework.

3.4 Why Legitimation Code Theory?

The choice of LCT is closely associated to broader intentions about what the study is trying to achieve. Perhaps because of the aforementioned false dichotomy between positivism and constructivism, within the field of education research there has been much debate about researching findings and their generalisability. Some may seek to conduct studies that produce what have been termed 'nomothetic' generalisations that generate rules or laws that are said to govern a particular population, and these are often conducted under the banner of positivistic research (Lincoln and Guba, 2009). This approach is problematic as it takes knowledge to be absolute in nature and decontextualised from culture and society and, as some argue, attempting to generate these sorts of findings in the field of education is somewhat misguided (Cronbach, 1975; Lincoln and Guba, 2009). This view of generalisation has pervaded much of education research to the extent that many only equate the word generalisation with this particular stance. However, in direct contrast with this approach, others may seek to make generalisations that are more 'ideographic', where the only intention is for readers to make their own interpretations from the findings and generalisations are limited to the direct context of the research data (Stake, 1978; Lincoln and Guba, 2009). The argument for this approach is that knowledge is experiential and therefore generalisations are only possible through individual interpretation of research findings (Stake, 1978). Some may argue that this should not be considered as generalisation at all, however it is merely generalisation at a much smaller scale to the nomothetic approach and, such arguments over the use of the word could be seen as a semantic issue, rather than a conceptual one. Nevertheless, this stance is also problematic as, by treating knowledge as simply a process of knowing in relation to personal experience, it may lead to the trap of knowledge blindness (Maton, 2014), something which this study attempts to avoid. As a caveat, it is not the case that any research methods commonly associated with either approach to generalisation should be discounted, rather they should be better understood with regards to the data they produce and what kind of generalisation this can be used for. Rather than opting for one or the other of these approaches, the intention in this study is to make generalisations that contribute to theory development; what Yin (2014: 40) describes as "analytical generalizations" and have been termed by others as the process of developing a "working hypothesis" (Lincoln and Guba, 2009: 38). This aligns with the social realist stance as it acknowledges the limited nature of any generalisation beyond the social and historical context of any one research study, but also suggests that findings from individual studies can contribute to knowledge building in a more generalised way. Put simply, the intention here is to contribute to knowledge by generating findings that are generalisable beyond the context of the study through use of a theoretical framework that is designed for this. In this way, it can be seen that a working hypothesis is generated which does not claim to be nomothetic in nature but also moves beyond the ideographic. Because of this, a well-defined theoretical framework was sought, particularly one that allows for there to be a clear and unambiguous relationship between the theory and the data. In the search for such a framework, I discovered LCT and was particularly drawn to the claims of being able to bridge the gap between theory and research data whilst avoiding the false dichotomy between positivism and constructivism, something that I felt had been missing within my research design until this point.

3.5 Legitimation Code Theory

This section will introduce Legitimation Code Theory by outlining its purpose, its relationship with Social Realism, and its theoretical roots. As a framework, LCT is concerned with explaining the basis of actions within social fields. It posits that underlying any “practices, dispositions and contexts” there are particular organising principles which are referred to as codes (Maton, 2016: 240). These codes are described in terms of legitimation, in other words, what an actor believes to be a legitimate way of acting in a particular context (Maton, 2014). Thus, the codes which form the basis of legitimate actions in a given scenario or context, regulate the ways in which desirable outcomes are achieved. In the context of this study, what this means is that the actions of teachers when teaching mathematics are governed by certain organising principles, or codes, and what the teacher sees as legitimate in terms of these is an important factor influencing the way in which they choose and use representations. Therefore, as well as being able to contribute to directly answering the research question, LCT will also help facilitate another goal of this study, which is to generalise to theory beyond the specific context, offering greater explanatory power.

Because Social Realism forms a part of the theoretical framework for this study, it is important to highlight the connection between Social Realism and LCT. Building on the work of Archer (1995), Maton (2016: 7) describes social realism as a meta-theory and LCT as an explanatory framework which maintains "dialogic relations" with meta-theories, whilst not being beholden to them. Separate from these two are the substantive theories generated by research studies (ibid., 2016). Although Maton (2014) claims that LCT is not beholden to any one meta-theoretical stance, it does appear that Maton's own Social Realist stance (Maton and Moore, 2009) is evident throughout LCT. This can be seen particularly in the claim made by Maton (2014, 2016) that LCT can overcome knowledge blindness and enable a multi-dimensional study of knowledge itself. This study adopts the meta-theoretical stance of social realism, using LCT as an explanatory framework to help connect the theory directly to the research data in the process of also generating substantive theory about the specific research questions.

Perhaps not surprisingly, given the strong connection to Social Realism, LCT is the result of Maton's (2014, 2016) attempts to extend the work of Bernstein and Bourdieu in creating a framework that does not start anew, but rather builds upon and advances their already significant and useful work (Maton, 2014). In particular, LCT can be seen to build upon both of their work in two significant ways. First, through LCT, Maton is aiming to avoid common false dichotomies, such as between positivist and interpretivist methodologies, that often prevent progress within sociological research, much in the same vein as Bourdieu (Grenfell, 2014; Robbins, 1991) and Bernstein (2000). The way in which Maton aims to achieve this is by creating theoretical tools that allow researchers to develop a direct connection between LCT and empirical data. By doing this, Maton claims that LCT avoids being overly theoretical and disconnected from practical realities, as well as avoiding empiricism, which can prevent any theoretical development that moves beyond the specific context of a particular study (Maton, 2016). In doing so, it appears that there is a direct connection to

Bourdieu, who strove to generate theory that was directly related to reality, thus helping bring people closer to the meaning of their actions (Grenfell, 2014: 15). Similarly, there is also a direct connection to Bernstein's (2000) idea of external languages discussed previously, where a mechanism by which a theory can maintain a close dialogic connection with real life data is of central importance. Maton claims that LCT can do this and therefore helps researchers to get "under the surface" of their data and explain the organising principles that lie behind it (ibid., 2016: 7).

Second, each of the concepts that together comprise LCT can be traced back to their roots in the work of Bernstein or Bourdieu. In particular, Bernstein's (2000) 'Code Theory' and Bourdieu's concepts of 'habitus', 'field' and 'capital' (Grenfell, 2014) can be seen as significantly influential on the design of LCT concepts. Bernstein's (2000) work is most visible within the LCT framework with Bourdieu's concepts acting as a more hidden foundation. This seems to be partly because Bernstein's theory provides a more structured, practical basis from which Maton could build the concepts within LCT, whereas the work of Bourdieu acts as more of a thinking tool, or "sociological eye" that is more intentional rather than operational (Maton, 2014: 19). However, Bourdieu's notion of 'gaze' is of particular importance, as LCT aims to help researchers develop a particular gaze when studying societal phenomenon, in other words, to help see the relational structures that underpin social activities; to see under the surface (Maton, 2014). In practical terms, this means that, to utilise LCT within research, it is important to have fully immersed yourself in the theory and its surrounding literature so that research data is analysed through the 'gaze' of LCT concepts.

The more operational aspects of LCT see Bernstein's (2000) code theory developed into what are referred to as legitimation codes which conceptualize the organising principles of practices, dispositions or contexts. In this way, the codes are not discipline specific and are designed to be able to be used to help explain the underlying "rules of the game" in any given social situation (Maton, 2016: 3). These codes together form the Legitimation Device, which is an extension of Bernstein's (2000: 25) notion of the "pedagogic device". Forming the Legitimation Device are different dimensions, each of which is designed to focus upon a narrow aspect of social practices and provides resources for studying different legitimation codes. There are currently five of these dimensions that together, comprise LCT: Specialization, Semantics, Autonomy, Temporality and Density (Maton, 2014). One of the qualities that makes LCT a relatively flexible framework is that each of these dimensions can be used on its own or in collaboration with any of the others. Use of the theory does not necessitate use of every single one of the dimensions; it is the problem situation, defined by the research question, that dictates which dimensions should be used (ibid., 2014). In this sense, LCT enables a creative approach to research that facilitates a central focus on real life issues; the framework is designed to work for the researcher for the purpose of enabling greater theory development. Although this is highly appealing as a researcher, one area of critique is the possibility that these five different dimensions are just five different theories in their own right. This poses the question of how well the five dimensions come together to form one cohesive theory, especially if researchers can simply pick and choose which elements they use. This study utilises two of the five dimensions and will aim to critically analyse the way in which they come together as parts of a single unified theory, rather than acting as separate entities. Because of this, it is not necessary to outline all the LCT dimensions here, instead only the ones that are pertinent to the research issue at hand will be discussed. This study is primarily concerned with teachers and their relationship to knowledge and beliefs, along with their actual classroom practices. The LCT dimensions of Specialization and Semantics suit these issues well and will therefore be drawn upon to form the primary way in which LCT will be used within this study. Alongside this, there is added benefit to using these two dimensions as they seem to have been utilised in other research studies to a greater extent than any of the others. So far, LCT has a steadily growing literature base with research studies that focus upon a range of social issues such as online learning (Chen, 2010), music education (Maton, 2014; Walton, 2020), science education (Ellery, 2017), initial teacher education (Macnaught et al., 2013; Walton and Rusznyak, 2020) and English literature education (Jackson, 2016). Although LCT has yet to be applied to the field of mathematics education, as a framework it offers a useful approach to consider the research question that this study seeks to answer. Nevertheless, the claims that LCT can avoid the false dichotomy between positivist and constructivist research, and successfully facilitate generalisation to theory that goes beyond the specific research context, is a bold one and one challenge within this study is to critically evaluate LCT and its application to mathematics education research.

3.6 LCT Dimension Characteristics

Although LCT dimensions are individually unique in focus, there are three important shared structural aspects that run across all LCT dimensions, and it is useful to briefly outline these areas before further discussion of Specialization and Semantics. First, as Maton (2016: 11) explains, “each dimension comprises a series of concepts centred on capturing a set of organizing principles underlying dispositions, practices and contexts.” What this means in practice is that each LCT dimension has a corresponding coding framework that has been developed from Bernstein’s (2000) code theory. Therefore, the specialization dimension generates specialization codes, and the semantic dimension generates semantic codes. Across different social situations, these codes may differ, and through the studying of these codes, Maton claims that the underlying ‘rules of the game’ can be better understood (ibid., 2016). Again, this study will seek to critically evaluate this claim. Specifically looking at how well these codes, that are manifested in different ways according to the social context, can come together to contribute to more general theory about legitimate social practices and the ‘rules of the game’. Second, because each dimension is not designed to be discipline specific, careful analysis and interpretation of each one considering a particular research focus is required to understand how they might be applied. In practice, this means that use of any dimension requires careful thought about how it applies to the specific context of any one study. To do this, specific examples that relate to the focus of this study will be provided when discussing the dimensions of Specialization and Semantics. Third, each dimension is built up of separate yet related concepts that are presented as continua that come together on a cartesian plane. Empirical data can be plotted onto any one of an infinite number of spaces within this plane to describe a code or collection of codes that enable greater explanatory power when compared to previous models such as Bernstein’s (2000) original code theory that presented codes in a more binary fashion (Maton, 2014). In this way LCT claims to facilitate a focus on the multi-dimensionality of knowledge and knowledge practices more so than previous frameworks. Figure 8 in the following section shows an example of such a cartesian plane. Nevertheless, a practical issue with this is the difficulty posed to researchers in being able to effectively communicate findings more widely. Within studies that have used LCT, various communication tools have been used such as plotting data onto a cartesian plane (Ellery, 2017) or showing the temporal movement of data on a line graph (Matruglio, Maton and Martin, 2013). Despite this, it seems to be difficult to clearly communicate the multi-dimensionality of knowledge practices once they have been identified within LCT research, and this will be another area in which this study aims to critically evaluate the effectiveness of LCT.

3.6.1 Specialization

The specialization dimension forms one aspect of the Legitimation Device and is the result of direct developments from Bernstein’s (2000: 99) concepts of “classification” and “framing”. Bernstein’s initial concepts refer to the idea that the boundaries between disciplines may be either strongly (+C) or weakly (-C) classified and that the framing, or control within these disciplines might also be strong (+F) or weak (-F). For example, in education the subject discipline of mathematics may be taught as a stand-alone subject with very few links drawn between it and other disciplines. Therefore, it may have clear boundaries drawn between it and other subjects within school. It also may be the case that it is taught in such a way that the teacher maintains high levels of control over the sequencing, pacing and criteria of knowledge to be learned. If this was the case, it would be an example of strong classification and framing (+C, +F). Maton (2014, 2016: 12) builds upon these concepts by highlighting the fact that, in any particular social situation, there might be stronger or weaker classification and framing of either knowledge practices, which he terms "epistemic relations", or of social relations to that knowledge, termed as "social relations". Although Bernstein touches upon these different aspects, Maton (2014) builds them into the LCT framework in an arguably more explicit and integrated way. To do this, the specialization dimension consists of two theoretical relationships, social relations (SR) and epistemic relations (ER), which come together to form the cartesian plane shown in figure 8.

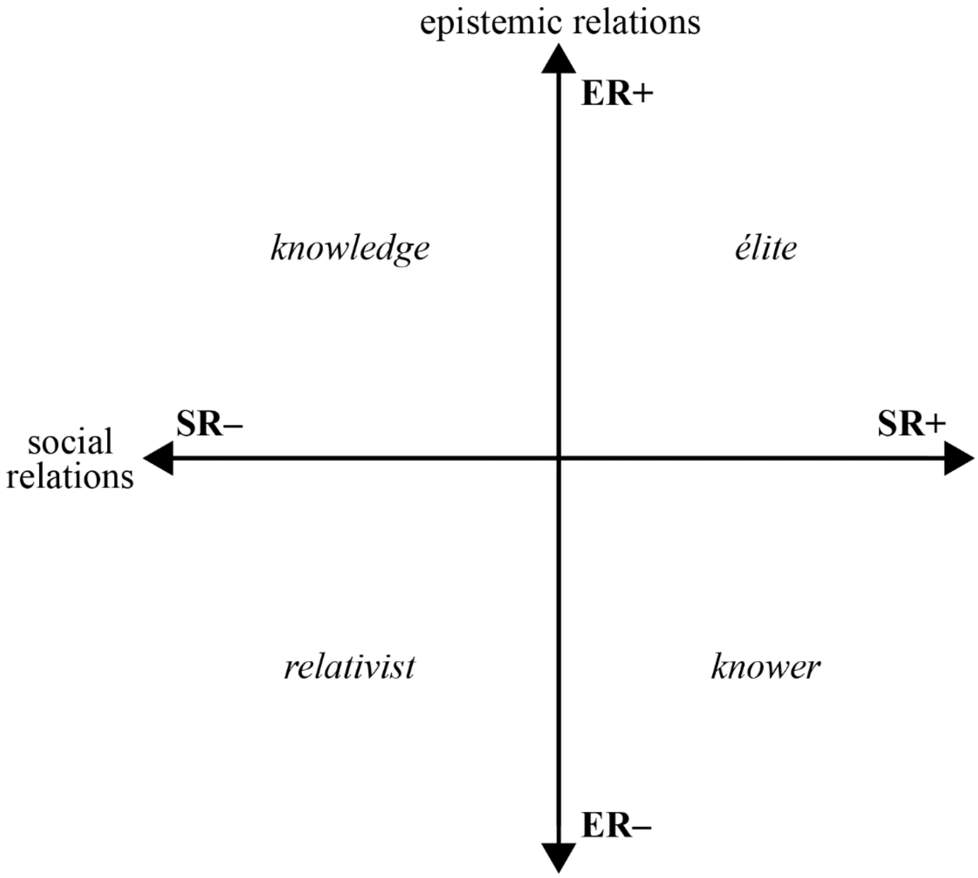


Figure 8 - The Specialization Plane (Maton, 2014: 30)

The specialization dimension draws attention to what can be legitimately claimed as knowledge (epistemic relations) and what kind of person is considered to be a legitimate knower (social relations). By placing these two relations together as a cartesian plane, four codes can be seen:

- *Knowledge codes* (ER+, SR-) where knowing particular specialised knowledge, principles or procedures is valued highly and the personal attributes of people are downplayed.

- *Knower codes* (ER-, SR+) where having any kind of specialised knowledge is not of great importance but having particular personal traits are highly valued. This might refer to anything from beliefs about natural born talent to socially developed attributes.
- *Elite codes* (ER+, SR+) where knowing particular specialised knowledge, principles or procedures and having the right sort of personal attributes are both considered to be highly important.
- *Relativist codes* (ER-, SR-) where neither specialised knowledge nor personal attributes are the basis of achievement. In Maton's (2016: 13) words – "anything goes".

(Adapted from Maton, 2016: 13)

The idea is that the cartesian plane can be used to show the nuances that lie within a set of data, rather than simply labelling something with one of the four codes. For example, a particular piece of data may be classified with an *elite* code yet be very close to the social relations line because the epistemic relation is only very slightly positive. In this case it would be a very fine line between coding something as *elite* or coding it as *knower*. For this reason, within this study, the development of a "translation device", which shows how the theory relates to the data is of significant importance (Maton and Chen, 2016: 43). Such a device will serve to provide empirical examples of how the varying degrees of strength within each code have been interpreted and will form part of the findings of this study. However, before moving on, there are a few important details to raise with regards to the specialization codes and their application within this study.

First, these codes are designed to analyse the *basis* by which achievement or success in a given situation is attained and not the focus of that situation (Maton, 2014). This is an important distinction to make and one that is specifically relevant to the analysis phase of this study. As an example of this distinction, it is highly likely in a primary school mathematics lesson, that a piece of specialised knowledge will be the focus of study (for example, adding two fractions). However, this does not mean that gaining or understanding that piece of knowledge is the *basis* of success in that situation. A teacher may well place low value on actual understanding of the specialised knowledge and high value on a specific personal attribute such as trying hard or demonstrating high levels of resilience (ER-, SR+), thus demonstrating a *knower* code. This is a particularly important point with regards to the way Specialization is used in this study when analysing data.

Second, Maton (2014) differentiates between different kinds of social relations, adding to the complexity of the dimension. Specifically, there appears to be a continuum of social relations, all of which would be considered 'SR+' within the specialization plane but that are quite significantly different. For example, some teachers might have a strong social relation code because they have a very rigid belief about fixed personal attributes: some pupils are naturally gifted at school maths, and some are not. This seems somewhat analogous to Carol Dweck's (2000) notion of a fixed mindset and something which has a significant research base related to mathematics in its own right (Boaler, 2016). In contrast to this, other teachers may also have a strong social relation code because they value the cultivation of certain personal dispositions such as having resilience, being curious or acting like a mathematician. Although both would be considered as strongly valuing social relations, one might be seen as detrimental within the mathematics classroom (seeing mathematical ability as a fixed personal trait) and the others may be seen as highly beneficial (being curious, having resilience and acting like a real mathematician) (Sun, 2015; Boaler, 2016). Therefore, when applying the specialization dimension within this study, it will be necessary to make a clear distinction as to what type of social relation is being emphasised within any of the data.

Third, there is a possibility that an aspect of Maton's (2014) chosen terminology within the specialization plane is somewhat value laden in itself. The so-called *elite* code brings with it a linguistic connection with ideas of elitism and elitist education which often come with negative connotations (Telling, 2020). It is not possible to say whether Maton uses this term with the intent of carrying with it an air of negativity or not, however within this study it is important to highlight that the term elite will be used without any relation to elitism in education. In fact, in the case of mathematics education, it may well be that teacher practices that fit into an *elite* code maintains a strong connection with what the literature would suggest is effective. For example, there is strong agreement within the literature that a successful mathematics lesson should involve the gaining of specialised knowledge as one of the keys to success (Sleep, 2009), however, it is also the case that developing personal attributes akin to that of a mathematician, such as curiosity and resilience, are also important (Schoenfeld, 1992; Sun, 2015; Boaler, 2016). Therefore, it might well be the case that, in some circumstances, an *elite* code of practice is an ideal for mathematics education. As a final note about terminology, in my study I use the spelling ‘specialization’ as opposed to ‘specialisation’ when referring specifically to the LCT dimension to maintain fidelity with the LCT literature. All other references to specialised knowledge will use the common English spelling.

3.6.2 Semantics

The semantic dimension offers another perspective of the Legitimation Device and is of relevance to this study because it focusses upon how, and what kind of, meaning is communicated. First, it is important to point out that, within LCT, Semantics is distinct from any general use of the term in linguistics. Where common linguistic interpretations tend to focus on it as the process of unpicking meaning, LCT focuses upon the complexity, referred to as “semantic density”, and context-dependence, referred to as “semantic gravity”, of meaning (Maton, 2016: 15; Wilmot, 2019). Again, this dimension is strongly influenced by Bernstein’s (2000: 157) work, in particular his theory of vertical and horizontal discourses which identifies forms of knowledge as either being highly segmented and context dependent (horizontal), or hierarchical and systematically structured thus able to stretch across many different contexts (vertical). Maton (2014) develops this further with the LCT concepts of semantic gravity and semantic density, which again can be seen as two continua that together form a cartesian plane (figure 9). Drawing links to Bernstein’s concepts, a knowledge form characterised by high levels of segmentation (horizontal discourse) is likely to display high levels of semantic gravity (being highly context dependent), however, it may or may not display high levels of complexity (semantic density). Therefore, by placing these codes onto a topological space to create four modes, Maton (2014) argues that the semantic dimension of LCT captures the multidimensional nature of knowledge forms and practices rather than treating them as binary, either-or, codes.

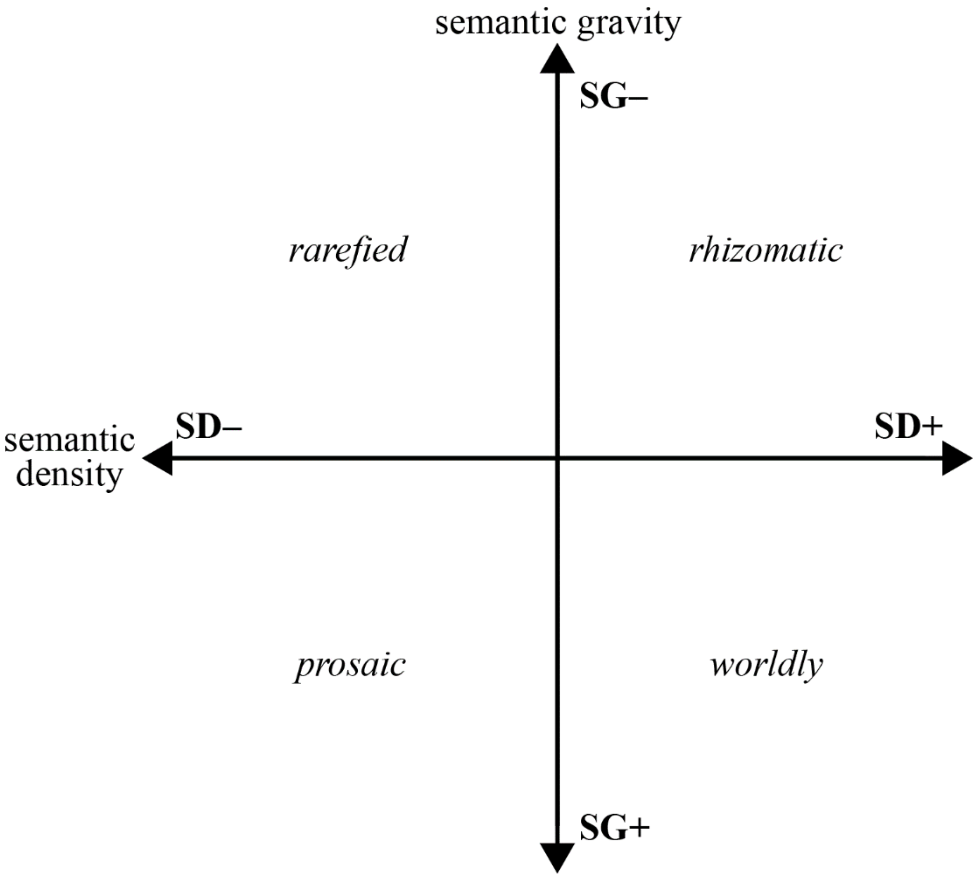


Figure 9 - The Semantic Plane (Maton, 2014: 131)

In the same way as the specialization plane, the semantic plane has four codes which shed light upon the semantic nature of knowledge forms and practices:

- *Rarefied codes* (SG-, SD-) identify when legitimacy is based upon context-independent meanings with relatively fewer meanings condensed together.

- *Worldly codes* (SG+, SD+) highlight where legitimate meaning is context-dependent yet has dense and complex interrelated connections.
- *Rhizomatic codes* (SG-, SD+) where meaning is context-independent and complex in nature.
- *Prosaic codes* (SG+, SD-) where legitimate meaning is relatively simplistic and dependent on its context.

(Adapted from Maton, 2016: 16)

As with specialization codes, these four codes that form the semantic plane are not designed to be either/or, rather they should be used in such a way that highlights nuances within data. Therefore, a translation device will also be created for this dimension so that there can be a clear connection between the data from this study and the theory of LCT (Maton and Chen, 2016). Nevertheless, some issues can be drawn out in advance of any data analysis as to how the content of this study may relate to the semantic dimension.

Because semantic codes are related to the communication of meaning, they bear direct relevance to the representation of mathematical objects. As discussed in the literature review chapter within the section about representation in mathematics teaching (section 2.3), mathematical objects can be represented in a wide variety of ways (Bruner, 1966; Sfard and Thompson, 1994; Duval, 2006). The way representations are used to negotiate meaning within a classroom setting is likely to be an influencing factor on semantic coding. First, take a common practice in schools which is to use realistic scenarios to represent numbers. As an example of this, figure 10 shows a representation being used to communicate meaning with a high level of semantic gravity and a low level of semantic density. The semantic gravity is high because the representation is highly context dependent. It is an image of a real-life scenario, flowers in a vase. The semantic density is low because there are no obvious connections with other mathematical objects, in fact the observer must see the mathematics in it by counting the flowers to even consider any mathematical meaning at all. Despite this, it would not take much for a teacher to use this same representation and increase the level of semantic density by asking a question such as ‘how many different ways can we figure out the total number of flowers?', prompting discussions about different ways to partition the number seven. Such a question invites the observer to see the image in a more mathematical way and suggests a higher level of semantic density by prompting the making of connections (e.g., a pupil may see seven flowers as 'two threes and one more'). Additionally, if the teacher then asked pupils to represent the flowers with plastic counters (figure 11), asking the question, 'Is this about flowers or is this just about the number seven?', the level of Semantic density is raised again because a connection is now being made to a new representation (counters). The semantic gravity here would also be lowered because a counter is slightly less context dependent and can be used to represent a wide range of things. This demonstrates how the social context, accompanying dialogue, and the role of the teacher are key factors in determining the semantic coding of the way representations are used.

Figure 10 - An example of a realistic scenario to represent the cardinal number 7

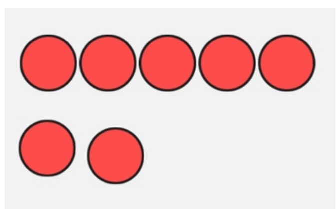


Figure 11 - Counters used to represent the seven flowers in figure 8

To further demonstrate the importance of the social influence of teachers and pupils, another example must be provided in the form of what is arguably the most used representation system in current times: the symbolic system used for mathematics in most countries around the world (Arabic numerals along with symbols such as '+' and '='). This representational register would be considered as having a very low level of semantic gravity, meaning that the symbols themselves bear very little relevance to any real-life applications. The number three, or ‘3’ written in this register, is simply a few marks on a piece of paper or screen that does nothing to hint at the possible real-life applications it may serve. Additionally, it could also be considered as having very high levels of semantic density, meaning that, to the person who has mastered an understanding of the number three, within the representation ‘3’ lies connections to a vast multitude of other areas of mathematics. This representational register that might be considered as the common language of mathematics, is a very semantically dense one. Nevertheless, as has been shown in the previous example, it is arguable that the amount of semantic density within the symbol ‘3’, or any other symbol from this register, is likely to be dependent on the way it is used by the teacher and understood by the pupils. When a teacher shows the symbol ‘3’ next to an image of three apples in front of a class of four- and five-year olds, it is likely that the semantic density of the symbol ‘3’ in such a context is quite low because the teacher will be using it in such a way to match pupils emerging understanding of numbers. However, on the opposite end of the spectrum, showing the number ‘3’ to a room full of teachers and asking what it might possibly be representing will elicit a broad range of responses demonstrating its potential semantic density.

3.7 Theoretical Framework Summary

To summarise the theoretical framework for this study, it is useful to consider it in three parts: social realism acting as a meta-theoretical backdrop, LCT acting as an explanatory framework and, the data instruments, along with the Specialization and Semantics LCT dimensions, acting as operational elements.

Social realism, as the meta-theory lying behind this study, is important as it frames the way in which knowledge claims will be treated in this study. Specifically, a methodological approach needs to align with social realism by avoiding the false dichotomy between constructivist and positivist research and attempting to study knowledge as real and known through social practices. This will then influence the type of knowledge claims that can be made in my study, which will aim to develop a working hypothesis that contributes to theory development, thus moving beyond the specific context of the research. To make this more than just an intentional claim, LCT has been adopted to act as an explanatory framework which seeks to enact these aims by making a more explicit connection between theory and data. LCT attempts to explain the organising principles that underpin any “practices, dispositions and contexts” (Maton, 2016: 240) and, in doing this, help explain the so called ‘rules of the game’. To make this an operational reality, LCT consists of five dimensions, two of which are adopted as part of this framework: Specialization and Semantics. The choice of these two dimensions directly relates to the research problem at hand. The final element of this framework sees the literature review chapter used to create data instruments. While these do not facilitate a direct dialogic relationship with the meta-theoretical backdrop of social realism, these are an important part of this study in that they provide useful guidance for method design and analysis. Used alongside LCT and its dimensions, the aim is that these data instruments will enhance the research and bring some important practical aspects to the findings so to maintain a degree of direct implication for teachers.

4 Methodology

The study of teachers' mathematical beliefs, knowledge and practice is complex, and a variety of methodological approaches have been adopted within previous research including mixed methods, ethnographic research, and case studies (Cobb, Yackel and Wood, 1992; Muis, 2004; Sleep and Eskelson, 2012; Sun, 2015). For this study, I adopted a case study approach in order to generate rich, in-depth data that would help un-pick the possible nuances surrounding the research question. The intention was to create a research design that provided sufficient explanatory power for me to contribute to theory, as well as to influence the policies and practices of schools and teachers. Underpinning this is the critical realist research philosophy that is closely related to important aspects of my theoretical framework.

This chapter will first provide an outline of critical realism as the essential philosophical basis for this study. Second, my position as an 'insider researcher' and the case study approach will be discussed, including some explanation of the choice of participant and how contextual factors affected this. Finally, the research methods and approach to analysis will be outlined, including discussion of ethical issues. As a reminder, the research questions are presented below.

Research question:

How do teachers' mathematical beliefs and knowledge influence their use of representations in the process of negotiating the mathematical meaning of fractions?

To fully answer this question, there are also four related questions that help focus the study:

1. *How can we effectively understand teachers' beliefs and knowledge*

about the nature of mathematics and mathematics education?

2. *How can we effectively understand how mathematical representations are used by teachers to communicate mathematical meaning in school maths lessons?*
3. *How does a textbook scheme influence teachers' beliefs and knowledge, and use of representations?*
4. *How can we explain the relationship between teacher beliefs and knowledge and the use of mathematical representations in the classroom?*

4.1 Critical Realism

Critical realism is a useful philosophical perspective that provides an alternative to the widespread constructivist and positivist paradigms often adopted within educational research (Danermark, et al., 2002, Olson, 2009; Scott, 2010; Denzin and Lincoln, 2011). Critical realism refutes the dichotomy that is often presented between quantitative and qualitative research, arguing for "methodological pluralism" that focuses more upon the intentions and needs of the research, rather than methodological ideology (Danermark, et al., 2002). In this way, critical realism can be seen as an "under-labourer" to social research, providing philosophical structure, without imposing ideological restraints (Joseph, 2002: 25). Importantly, it is also a philosophical underpinning to social realism, therefore its adoption here is congruent with the theoretical framework of this study, thus strengthening the overall design of my study and ensuring coherence (Danermark, et al., 2002; Maton, 2014; Wheelahan, 2010; Moore, 2013). Nevertheless, it is not a homogenous movement and many different critical realist perspectives can be found within the literature (Danermark, et al., 2002; Roberts, 2014; Fletcher, 2017). Therefore, it is important to outline what aspects of critical realism are pertinent to this study and to describe how they permeate the research design. This section discusses the key critical realist concepts of ontological realism and epistemological relativism, ontological depth, and *retroduction*. These will be introduced below and then referred back to at points throughout this chapter in order to demonstrate how they have influenced the research design.

First, in delineating between the ontological and epistemological, critical realism is both *realist*, because it emphasises the real existence of knowledge external to human perception, and *critical*, because it acknowledges the fallible nature of knowing (Scott, 2010). This distinction can be traced back to Roy Bhaskar who argued that the nature of things is *intransitive* (ontological realism) and the way things are known is *transitive* (epistemological relativism) (Archer et al., 1998; Scott, 2010; Bhaskar, 2011). Ontological realism suggests that there is a reality that our knowledge refers to and that this is something more than just relativistic standpoints. In other words, there is an independent reality, existing beyond discourse, that shapes knowledge of the world (Maton, 2014). Bhaskar (1998: 16) provides a helpful demonstration of this type of intransitive knowledge: “If men [sic] ceased to exist sound would continue to travel and heavy bodies fall to the earth in exactly the same way”. This is distinct from epistemological relativism which asserts that, as humans, we can only know the world through social interaction or socially produced outputs, which are historically situated and subject to cultural influence (Maton, 2014). Even when sociological research is conducted under the most rigid ‘scientific’ controls, it can be seen that researcher influence plays a significant role (Kelly, 2006). In this sense, this knowledge is transitive because it may change over time and consists of the outputs created by humans that help us understand knowledge of the world (Bhaskar, 1998). It is through these key concepts that critical realism departs from positivism and constructivism (Collier, 1994; Fletcher, 2017). It is argued that both positivism and constructivism suffer from the “epistemic fallacy”, where reality is reduced to what can be empirically known (Scott, 2010: 17). This is one argument for why the dichotomy between positivist and constructivist research is false and relates to social realist arguments of “knowledge blindness” where the nature of reality is overlooked because it is reduced to ways of knowing (Maton, 2014: 4). In the case of positivist research this is through knowledge being treated as a “container” of reality (Fletcher, 2017: 182). In other words, there is a conflation of knowledge and reality itself; ways of knowing are treated as providing direct access to reality. In the case of constructivist research, it is through knowledge being treated as a lens, or standpoint, towards reality that knowledge blindness occurs (ibid., 2017). By treating knowledge in this way, attention may be paid to ways of knowing rather than the object of knowledge itself. Critical realism claims to avoid this epistemic fallacy in a number of ways, two of which are particularly pertinent to this study – by acknowledging and accounting for ontological depth and through the application of *retroduction*.

The second concept within critical realism that is of central importance here is 'ontological depth'. Ontological depth points to the fact that reality is highly complex and has multiple layers of meaning (Olsen, 2009). Some have used an analogy of layers or concentric circles to explain this, where the inner circle would be the psychological realm, followed by the body and then the social realm and this would continue to include other areas such as the chemical, physical and astronomical realms (Neff and Olsen, 2007). Important to the idea of ontological depth is the fact that these realms are not separate but are all interconnected and permeate one another (ibid., 2007). For example, the reality of a mathematical idea such as the number '3' could be seen to reside in the psychological realm, yet it has been developed and understood as an idea through interaction and establishment of norms within the social realm and can also be represented in the physical realm. Therefore, the layers or concentric circles analogy of ontological depth is an over-simplification of the concept and has been criticised for applying limited human concepts such as spatial language (inner/outer, upward/downward etc.) to a highly complex philosophical idea, as has been outlined in further detail by Martin and Heil (1999). Ontological depth has important practical implications for this study in that it suggests that social reality, or social knowledge objects, are the real manifestations of idealised knowledge objects that have been defined through social discourse, yet these ideas are much more complex and multi-layered than can be relayed through discourse itself (Scott, 2010). Simply put, the nature of knowledge, when it is being studied in real life, is highly complex and nuanced and can be known in many different ways (Olsen, 2009). What this means in practice is that careful consideration must be put towards research methodology, in particular, how the multi-layered nature of knowledge can be captured through research design. For example, Olsen (2009) suggests that research utilising individual preference data, for example from surveys (a common approach within the mathematics beliefs literature), is unlikely to capture ontological depth as it only accounts for a single layer and perspective of a social phenomenon. This is one of the reasons why I adopted a case study of one teacher, collecting multiple sources of data about a particular phenomenon from different perspectives. Specifically, this study uses a range of methods offering multiple perspectives, alongside a multi-layered approach to data analysis providing yet another layer of understanding. Part of this involves the key critical realist concept of *retroduction*.

Retroduction is one of the more practical methodological tools proposed by critical realist philosophy and involves looking for underlying patterns that can explain why themes are occurring within empirical data (Danermark, et al., 2002; Olsen, 2007; Belfrage and Hauf, 2017). It is a form of analytical reasoning that can be put alongside more traditional forms such as induction and deduction, although it is also distinct from these in that it claims to move beyond what is empirically observable (Danermark, et al., 2002; Olsen, 2007). By utilising *retroduction* the researcher is prompted to look beyond empirically identifiable themes within the data, towards explanations for why these themes are there. In short, *retroduction* requires explanations for explanations (Scott, 2010). In this sense, it is another way of accounting for ontological depth because it is assumed that social reality has underlying patterns that can be known in different ways, and *retroduction* as a form of analytical reasoning can support this process (Olsen, 2007). Within this study, the process of *retroduction* is built into the approach to data analysis, discussed later in this chapter, but also taken into account within the theoretical framework through the adoption of LCT, which itself has been designed as an explanatory framework that claims to be able to aid researchers in getting “under the surface” of empirical data (Maton, 2016: 7) and help with providing explanations for explanations, as is the case with *retroduction* (Danermark, et al., 2002; Olsen, 2007; Belfrage and Hauf, 2017).

4.2 Researcher Positionality and Reflexivity

Bearing in mind the central concept of epistemological relativism within critical realism, it is important to recognise my position as the researcher and how this is likely to influence the research process. There are two important points to highlight here – my stance on the co-creation of knowledge, and my position as an ‘insider’ researcher, including my role within the English education system. In this section I will outline the approaches taken to help maintain a self-reflexive stance throughout my study, which include the use of ‘member checking’ during data analysis (Lincoln and Guba, 1985; Doyle, 2007), as well as the use of a reflective journal throughout the study (Watt, 2007). Both contribute to the reflexive approach taken here, which I take to mean how my own experiences and context, which may be fluid and develop throughout, inform the research process and outcomes of the study (Etherington, 2004; May and Perry, 2011).

4.2.1 Co-creation of Knowledge

Partly because of the notion of epistemological relativism outlined in the previous section, and partly because of my desire to maintain high levels of reflexivity, I take the research process as one which involves the co-construction of transitive knowledge between myself and the participant. Doing this required me to have a deep understanding of my own positionality, as well as understanding that of the research participant. Arguably, this is one way in which the research design accounts for the notion of ontological depth (Olsen, 2009; Scott, 2010). By acknowledging the perspectives of myself and the research participant, I am more likely to understand the nuances and layers of the knowledge created, ensuring that the knowledge output from this study is socially robust (Nowotny, Scott and Gibbons, 2001). However, although such an approach was taken, this does not mean there was any deliberate manipulation of participant behaviour during the process. This study takes an explanatory approach, attempting to better understand the beliefs, knowledge, and practices of the research participant through direct observation alongside interviews and textbook analysis. One of the central ways in which I enabled this stance to become a reality during the process was through conducting 'member checking' during the data analysis (Lincoln and Guba, 1985; Doyle, 2007). In basic terms, this process can be described as "taking data and interpretations back to participants in the study so they can confirm the credibility of the information and narrative account" (Creswell and Miller, 2000: 127). Nevertheless, this description seems overly simplistic and, in my study, the act of liaising with the research participant and going over initial data analysis proved to be a complex yet important process. The complexities around validity, trustworthiness and ethics when using member checking are captured well by Hallet (2012), who argues that it is a valuable process but that researchers must use it with care, avoiding using it in a mechanical way and instead engaging critically with the participant in a meaningful conversation. Within my study, the process proved to be invaluable as it helped me see where my initial analysis was perhaps skewed towards my own position with regards to mathematics teaching and helped me better enhance the voice of the research participant. As an example of this, during the teacher problem tasks interview (section 4.7) Gillian made a mistake when calculating the division of two fractions. Initially, I had interpreted this as a deficit in her understanding of this mathematical process and started to think about what the implications of this were in terms of her belief and knowledge system. However, upon engaging in the member checking process we were able to have a further discussion about this where Gillian confirmed this to be something she knew about in a deep way (and she was able to demonstrate this during the discussion), however it was the unexpected nature of it appearing within the problem task that had thrown her. Additionally, she was able to explain that, because dividing a fraction by another fraction was not something she taught regularly (due to it being excluded from the primary curriculum content she was following) it was therefore something she was a little out of practice doing or thinking about. This caused me to reflect on how I was influenced by my own experiences working with some secondary schools (who teach this regularly) which had led me to consider this aspect of working with fractions as something that all teachers should have at the forefront of their minds. Although this is just one example, it does illustrate the wider way in which the member checking process helped me develop co-created knowledge. In particular, how it supported me in reflecting upon my own knowledge, beliefs and experiences and consider the way in which these were influencing my data analysis.

4.2.2 My Position as an 'Insider' Researcher

Alongside understanding my stance on the co-creation of knowledge, to maintain high levels of reflexivity it is also important to acknowledge my position as an 'insider' researcher within this study (Dwyer and Buckle, 2009). This involves articulating how my role as a researcher interacts with my professional role within the English education system, and how I balanced the two positions throughout my study. My professional role involves working for a multi-school group comprised of both primary and secondary schools, as well as leading a 'Maths Hub' (a large regional organisation linking approximately 300 schools), which receives funding from the UK government to support mathematics education reforms. Prior to this role, I also spent considerable time working in both primary and secondary schools supporting the teaching of school maths as a specialist teacher. This means that I have direct knowledge of the reform-oriented work happening in schools and an indirect professional relationship with the research participant who had taken part in professional development provided by the Maths Hub that I lead. Nevertheless, it also means that in some ways I am a government policy mediator within a system that some have conceptualised as "state-market assemblage" (Boylan and Adams, 2023: 3). What is meant by this is that there is a tension between the promotion of school autonomy and potentially increasing state control over the content of professional development initiatives (ibid., 2023). Within English education there has been an increase in system leadership, a concept articulated and championed by Michael Fullan (2004), which is often described as providing greater autonomy for schools through devolving the leadership of educational initiatives to those working within the school system (Hopkins, 2009). My role as a Maths Hub lead may be seen as one form of this system leadership. However, it is argued that behind this veneer of increased school and teacher autonomy is an undercurrent of state influence over the content of professional learning, particularly in relation to 'teaching for mastery' and this is what is referred to as ‘state-market assemblage’ (Boylan and Adams, 2023). With regards to my position within my research study, this had several implications. In Bordieusian terms, my insider role means that I arguably have a pre-developed ‘gaze’ for the object of my study (Robbins, 1991), although one challenge was to develop this gaze further when analysing the data from different perspectives within my theoretical framework. Additionally, because of my professional role, I spend a lot of time in other teacher’s classrooms, therefore the process of observing lessons in a critically analytical way was one that was easy to acclimatize to. Nevertheless, although I do not hold any official position of authority over the participant, my leading role within our geographical region is likely to have some influence on the research process. For example, the research participant teacher knew me and about my professional role, and I considered that this might be expected to create some caution and perhaps a tendency to be compliant, towards what they perceive to be my views and towards the national ‘teaching for mastery’ policy itself. However, the main strategy of the national implementation process of ‘teaching for mastery’ (especially within my local Maths Hub community) is collaborative professional inquiry, adopting a critical perspective towards the policy, in terms of its interpretation, its implications for classroom practice, and its impact on learning. I considered that this characteristic of the policy implementation helped to balance the risk of compliance. The data generation methods, including observation, video, gathering evidence of pupil work, and professional conversations to review teaching and its impact on learning, were all closely aligned to the policy implementation process and methods. Therefore, within the over-arching structural position of policy implementation, collaborative professional inquiry provided some balance of professional autonomy for collaboration between the teacher participant and me as researcher. Nevertheless, this situation necessitated high levels of reflexivity, always bearing in mind my working relationship with the teacher participant in relation to the national context. This also raised ethical considerations which will be discussed in a subsequent ethics section (section 4.9). Therefore, I needed to maintain some form of differentiation between my professional role and my role as a researcher and this meant continual reflection on my position and personal thoughts about mathematics education within England. To help me do this, I utilised a reflective journal (Watt, 2007), based upon the notion of writing as a "method of inquiry" to find out more about yourself (Richardson, 2000: 923). This was a habit which I had begun before embarking upon this study, therefore I was able to continue in this practice rather than starting anew. The physical journal that I used was a small notebook that I could always keep with me and use to write down thoughts and reflections that were occurring to me both during the research process, but also during experiences in my professional role. For example, looking back through the journal I could find instances where I had been attending meetings or workshops that had caused me to reflect on my stance towards my research study. This enabled me to see how both of my roles were interacting but also helped maintain a critical distance between the two. One example of this is an entry from the same time at which I was beginning data analysis, where I had been taking part in a development session with teachers where they were discussing the 'CPA' approach to representations (Merttens, 2012). The specific journal entry is a reflective monologue about the 'CPA' approach, what these teachers where discussing and what I had been thinking about in relation the analysis using the semantic dimension of LCT. This sort of reflective journal entry helped me better understand how my professional role as a system leader was interacting with my role as a researcher and helped me ensure high levels of reflexivity.

Having outlined critical realism as the philosophical backdrop to my study and explained my approach to researcher positionality and reflexivity, it is now important to define my approach to case study design, alongside the methods of data collection and approach to analysis.

4.3 Case Study Design

Case study methodology is multifaceted in that there are many different perspectives on what constitutes a case study and how it should be done (Yin, 2006; Yazan, 2015; Cohen, Manion and Morrison, 2017). Nevertheless, there is widespread agreement that a case study can be seen as the study of a particular case (although the determining of what can be considered as a 'case' is contested in itself) for the purpose of describing and explaining a particular phenomenon (Bassey, 1999; Stake, 1978; Swanborn, 2010; Yin, 2014; Cohen, Manion and Morrison, 2017). This study focusses upon the phenomenon that is teachers' beliefs and knowledge and their use of representations when teaching fractions, and the case itself is one primary school teacher. In this section, I will first outline why a case study approach has been chosen and second, I will describe the specific case study design that has been adopted, including discussion of some contextual factors that influenced this.

The rationale behind opting for a case study approach in this study is twofold and relates to the research question and the research paradigm. First, it is widely attested that case study research is particularly powerful when in-depth understanding of social phenomenon is sought (Stake, 1979; Yin, 2014, Gillham, 2000). Additionally, Stake (1979: 19) highlights the fact that case studies tend to be "in harmony with readers' experience", drawing attention to the way in which people often make sense of the world by looking at specific cases of a phenomenon. Because this study is primarily focussed on developing better understanding of a highly complex social phenomenon, and also has the secondary aim of producing findings that will help teachers understand how this may impact upon their own practice, then a case study methodology is appropriate. Nevertheless, case study as a research approach has been criticised by others due to concerns over generalisability of findings, researcher bias and their tendency to embalm practices that in reality are always changing (Bassey, 1999; Creswell, 2014; Cohen, Manion and Morrison, 2017). Nevertheless, it is arguable that these criticisms depend on intentions behind making generalisations along with case study design and conduct, rather than being implicit flaws with the approach (Yin, 2014). Where there have been concerns over the generalisability of case study findings, these appear to be focused upon nomothetic generalisations (Bassey, 1999; Lincoln and Guba, 2009). As outlined in the theoretical framework chapter, it is not my intention to make nomothetic generalisations, therefore it will be important that any findings clearly state that they are part of a working hypothesis (Lincoln and Guba, 2009). Second, a key aspect of case study design is that multiple data sources are collected around a single point (Yin, 2014). This will help contribute to a research design that accounts for the critical realist notion of ontological depth (Olsen, 2009). In practical terms, what this means is that I acknowledge the complexity and multi-layered nature of the phenomenon I am studying and have attempted to design a case study that will capture it from multiple perspectives. This prompts a discussion of how my case study was designed, focussing on three key issues – defining the 'case' and explaining the contextual influences upon this, planning for data collection, and choosing the participant.

Although there are key differences between the way authors define what is, and isn't, a 'case' within the literature, there is firm agreement that, whatever approach you take, you must be clear about defining the boundaries around the case you have chosen (Stake, 1979; Yin, 2014, Yazan, 2015). In this study there are a number of possibilities – for example I could focus on a whole school, or a teacher as an individual, or a teacher and one class together, or even the teacher and all of the classes they teach. Each one of these has potentially useful aspects to offer however, because the research question is firmly rooted in the knowledge, beliefs and practices of the *teacher*, then I decided that it must be the teacher who is the 'case' to be studied. Nevertheless, because the phenomenon being studied is also a social one it is not possible to take them as an individual in isolation. As Gillham (2000) highlights, cases often merge into their contexts making precise boundaries hard to draw. Therefore, the context within which they work (the classes they teach, their school and the wider school system) will be carefully considered throughout the study and any methods deployed will take this into account. For example, contextual details ranging from the size of the school and class taught, to the year group and typicality of the pupils compared to the teacher's prior experience, will be gathered. Additionally, the design of the data collection methods accounts for significant contextualisation by using stimulated recall interviews combined with lesson observation and lesson videos, thus grounding my study in classroom practice.

Of key importance to the design of my study was the decision to focus on one teacher. I had initially intended on conducting a *multiple* case study, where data would be collected from four individual teachers, comprising four distinct cases. The rationale for conducting a *multiple* case study was to produce greater variation within the data. Nevertheless, two developments led me to adapt this approach. First, and most importantly, develoment of my theoretical framework through the integration of Legitimation Code Theory (Maton, 2014) led me to realise that I did not need a *multiple* case study to generate data rich enough to answer my research question. Although it may have generated greater variation within the data corpus, I would not have been able to go into such depth and apply LCT so effectively through the retroductive phase of my analysis (section 4.8) had I collected data from more than one teacher. Second, school closures in the Summer term and restrictions regarding travel and entering schools (due to the Coronavirus pandemic) meant that I was not able to do any further data collection beyond that of my first case, which I had collected before the pandemic began. This meant that I had to re-think my initial idea of conducting a *multiple* case study. As an 'insider' researcher myself, I knew firsthand what it was like in schools and could see that it would be a long time before in-school data collection of the sort I was proposing would be able to take place. Despite having to re-evaluate my options due to the pandemic, I was ultimately glad of the interruption it caused me with regards to my study. It prompted me to take the time to consider what sort of data analysis would lead to answering my research question most effectively and helped me avoid making the mistake of conducting a multiple case study. It steered me towards a better design, by introducing LCT, which answered the research question in a deeper and more productive way.

Bearing this in mind, one of the key reasons that my single case study produced rich enough data to answer the research question, was due to the carefully designed data collection methods. How rigid an approach to planning for data collection should be is something which authors seem to disagree on when it comes to case study design (Yazan, 2015). Where Yin (2014) argues for rigid planning allowing for very little deviation or development during data collection, Stake (1979) takes the opposite approach arguing that you cannot plan much at all as you do not know what you are going to be required to collect until you have experienced the case. My case study design took influence from both authors and took the middle ground, utilising stimulated recall interviews, belief and knowledge tasks for teachers, classroom observations and associated videos of these lessons, document analysis of teacher and pupil output from lessons, and a textbook analysis. The data collection process was carefully planned so that I would collect data pertinent and rich enough to answer the research question, however I also added additional aspects once I had already begun to collect data. Initially, I had planned to collect data on three different occasions – the teacher problem tasks followed by two separate lesson observation and stimulated recall interviews. Even within this planned design, I was able to make small changes between data collection points however, after these three had been completed, during early data analysis I recognised the need for some follow up data collection and thus adapted the plan to include a follow up interview and then the textbook analysis. Table 8 demonstrates how I had planned for data collection methods to pair with the specific guiding research questions of this study, which steer the data collection towards gathering pertinent data for the main research question. This includes any additions made once the data collection was underway in italics.

Table 8 - Research questions and related data collection methods

Research question	Research methods *(those added at a later stage, in response to early analysis, are shown in italics)*
1. How can we effectively understand teachers' beliefs and knowledge about the nature of mathematics and mathematics education?	• Pencil and paper belief and knowledge tasks for teachers with associated interview • Semi-structured teacher interviews with stimulated recall • *A follow up interview was conducted after the initial data collection to ask additional questions about this particular question*
2. How can we effectively understand how mathematical representations are used by teachers to communicate mathematical meaning in school maths lessons?	• Video recorded lesson observation • Semi-structured teacher interview with stimulated recall • *Analysis of the teachers' textbook (added at a later stage during data collection)*
3. How does a textbook scheme influence teachers' beliefs and knowledge, and use of representations?	• Semi-structured teacher interviews with stimulated recall • Document analysis of planning materials and pupil output from lessons • *Analysis of the teachers' textbook (added at a later stage during data collection)*

4. How can we explain the relationship between teacher beliefs and knowledge and the use of mathematical representations in the classroom?	• Analysis of all case data

Finally, it is important to outline how the teacher for this case study was chosen. Because I initially set out to conduct a multiple case study, I had intended on having four participants who matched the criteria set out in table 9.

Table 9 - The initial criteria for purposive sampling of case study participants

Case	Uses a Textbook as Main Source of Lesson Content	Has Some Type of Mathematics Specialism	Age of Pupils
Case 1	Yes	Yes	Key Stage 2 (7-11 years)
Case 2	Yes	No	Key Stage 2 (7-11 years)
Case 3	No	Yes	Key Stage 2 (7-11 years)
Case 4	No	No	Key Stage 2 (7-11 years)

In reducing the number of cases to one, I chose a participant who met the first criteria of ‘using a textbook and having some type of mathematics specialism’. The reason being that meeting these criteria were the most important. First, the use of textbooks is integral to this study because of their increasing use within England and because of the support they have received from the government (section 1.2). I have theorised that the way a textbook represents fractions is likely to play an influential role in teachers’ beliefs, knowledge and practices relating to representation use (section 2.4) and therefore selecting a teacher who was using a textbook was on key importance. Second, it is likely that a teacher who has a greater degree of specialism in mathematics may well have better knowledge of fractions, and their associated representations, compared to one without such specialism. In my study, I consider ‘some type of mathematics specialism’ to refer to primary teachers that meet one of the following criteria:

- They have an undergraduate degree in a mathematics related subject

- Their teacher training included a specialism in mathematics
- They are working as a system leader who specializes in mathematics (Specialist Leaders in Education, Maths Hub Mastery Specialists)
- They are a mathematics subject leader in a school with at least three years' experience in the role

Within the next four sections, I will outline the specific data collection methods that together formed the case study, and how these were utilised to ensure the generation of data pertinent to the research question.

4.4 Semi-structured Interviews

To gather data about most of the research questions I used semi-structured interviews, two of which also included the use of a stimulated recall approach, which is becoming increasingly popular in the fields of education, medicine, and psychology (Pirie, 1996; Lyle, 2003; Rowe, 2009; Robson, 2011; Welink et al., 2020). Semi-structured interviews are a popular tool within case study research, primarily because they enable the researcher to gather rich data about specific topics whilst still providing the participant with a degree of choice about what they want to talk about (Rapley, 2007; Yin, 2014). I decided to utilise a stimulated recall approach to two of the interviews in my study because this allowed me to gather more contextualised, closer-to-practice data about teacher knowledge and beliefs (Lyle, 2003). In total, I conducted four interviews within my study. The first was an interview related to the 'teacher problem tasks', which I outline in detail subsequently in this chapter (section 4.7), the second and third utilised the stimulated recall approach and the fourth involved asking follow-up questions and conducting member checking (Lincoln and Guba, 1985; Doyle, 2007), after some initial analysis had taken place. The process of member checking (see section 4.2.1) was conducted by online conference meeting due to the disruption caused by the Coronavirus pandemic. This had led to school closures and travel restrictions that meant I was no longer able to conduct further data collection of this sort in-person. However, both the participant and I had become familiar with online conference meetings, and it worked well. On reflection, my approach to interviews (especially the teacher problem tasks) would not have worked in this format due to the challenging nature of the interview content. Nevertheless, for this member checking aspect it was effective as I could share my screen, within initial analytical notes for us to discuss.

As is suggested within the literature, each interview was guided by a series of questions and prompts whilst still allowing time for follow-up questions and discussion (Robson, 2011). The first interview involved completion of the 'teacher problem tasks' that I have designed for this study (section 4.7) and the ongoing discussion as these were being completed was audio recorded and then transcribed. The second and third interviews started with some general discussion about mathematical knowledge and beliefs and then became more focused upon the lesson video, allowing the participant to use the fast forward and pause controls to go through the lesson and reflect upon it. Again, these interviews were also audio recorded and then transcribed. The fourth and final interview took place after I had conducted some initial analysis and involved asking some follow up questions to clarify specific things alongside providing the opportunity for member checking of the data. This meant that my analysis of the interview data was tested out with the participant so that I could gather further data about their perception of my interpretation, thus enhancing the co-constructed nature of my analysis and contributing to my self-reflexive stance (see section 4.2). I have included the interview prompts used in the final three interviews in the appendix (appendix 1) as this helps provide some context when reading my analysis of this data in subsequent chapters.

Nevertheless, despite being a fundamental research tool, interviews are not without their weaknesses (Denscombe, 2003). The "interviewer effect" (Gomm, 2008: 221) refers to the way the researcher's own beliefs, assumptions and pre-conceptions may influence the data collected during the interview. Some attempt to reduce this as much as possible, taking an almost clinical approach, so that data can be analysed without consideration of who conducted the interview (ibid.., 2008). However, in my study, I take the stance that this is not possible and even with measures in place to reduce interviewer influence on the data, I acknowledge that interview data is co-constructed between the interviewer and the interviewee (Holstein and Gubrium, 1994). In line with others, I argue that it is not possible for the researcher to 'contaminate' interview data when they have been involved in the very creation of that data themselves (Gubrium and Holstein, 2002). This has important implications for analysis and the interview data must be interpreted acknowledging the context within which it was generated, rather than as providing direct access to any participant's experienced reality (Rapley, 2007). In particular, it is important that I maintain high levels of reflexivity throughout the research process so that I am aware and acknowledge my own influence upon the data (Etherington, 2004). This also aligns with the critical realist notion of epistemological relativism because it acknowledges that interview data as a way of knowing is historically situated and subject to social and cultural influence (Maton, 2014).

As well as issues concerning the effect of the interviewer and reflexivity, using a stimulated recall approach within interviews also requires careful planning (Lyle, 2003; Rowe, 2009). Within my study, each stimulated recall interview started with some general discussion about mathematical knowledge and beliefs and then became more focused upon the lesson video, allowing the participant to use the fast forward and pause controls to go through the lesson. This presented an alternative way to reflect on practice and unlike in the classroom context, there was time to relive experiences and watch things multiple times (Sherin and Han, 2004). Additionally, it was left open as to who could pause the video and choose what to discuss and this contributed to the co-constructed nature of these interviews (Rowe, 2009). This was a method that I had used in previous research and found to be particularly powerful (Boyd and Ash, 2018b). In this way, the study was designed to gather data on closely contextualised knowledge and beliefs whilst providing an opportunity for participants to discuss how this relates to more globalised knowledge and beliefs about mathematics and mathematics teaching. Nevertheless, although the approach has often been used without critique by researchers in education, it is important to consider the sort of data intending to be gathered and the time between event and simulated recall interview (Lyle, 2003). Simulated recall has been critiqued as an approach to gathering data about concurrent cognition (i.e. what a person is thinking *as* they are doing something) because the data is more likely to provide insight into what the person is thinking *about* what they *were* doing as they react to the video (ibid., 2003). In essence, asking participants to watch video of an event back, after the fact, provides an opportunity for them to reflect on that event and see it from a different visual perspective (unless the video is shot from their point of view), and it is important for researchers to be aware that this is the nature of the data being gathered. This was of benefit to my study because I wanted to gather data about teacher beliefs and knowledge and not about one teaching instance. Therefore, using the technique to prompt reflection and further discussion provided me with richer data about knowledge and beliefs in a more global way, rather than just about one specific situation (Reitano, 2005). In addition to this, another more practical issue raised by Lyle (2003) is that the simulated recall interview should ideally take place shortly after the event itself to avoid any tacit or short-term memory of the event being lost during the simulated recall process. The design of my study accounted for this by conducting the simulated recall interview on the same day as the event itself no more than one hour after each lesson. In practice this meant that the process of going through the lesson video and discussing it was easier as both me and the participant could remember and focus in on specific aspects of the lesson we wanted to discuss.

4.5 Lesson Observations

Although a large majority of the data was gathered through interviews, to fully answer the research question, it was important to conduct some observation of what the teacher did in the classroom. This is primarily because of the notorious discrepancy between people’s espoused beliefs and their enacted beliefs, as has been found in previous studies (Erikson, 1993; Raymon, 1997; Philipp, 2007). The critical realist concept of ontological depth also requires a multifaceted approach to data collection and adds a second reason for why lesson observation was important.

Despite this, direct observations are not without drawbacks themselves (Brown and Dowling, 1998). Although some support the use of observation and argue that they facilitate a focus on “what actually happens” (Burton, Brundrett and Jones, 2008: 73), within this study I am aiming to acknowledge that an observation will simply provide multiple perspectives of an event, not direct access to reality. By considering my own version of events, along with the teacher’s version, by using stimulated recall, I am aiming to create rich data that can provide insight into the research question. Here it is important to consider the phenomenon of “reactivity” which acknowledges that, simply by entering the situation, the researcher will affect things in some way and participants being observed may change their behaviour from what is normal (Robson and McCartan, 2016: 320). Taking this into account, three core strategies were adopted to try and reduce the amount that this might weaken the design of the study. First, as with the teacher interviews, maintaining high levels of reflexivity throughout was important, acknowledging and reflecting on my own influence on the data. Second, I used a “marginal participant” approach where the observer does not take an active role in any lessons observed but their presence is acknowledged by all in the room (Robson, 2011: 323). Arguably this helped reduce any reactivity as this is a normal role for someone like me to take when in a classroom with teachers – it is not something that should be at odds or surprising to any teachers or pupils. Finally, I also used the technique of “habituation” (Brown and Dowling, 1998: 47). This is where the researcher enters the setting before the actual observation to ensure that those being observed are somewhat familiar with their presence. This was particularly important in helping pupils get used to having an iPad filming at the back of the classroom, something which they were keenly aware of during the trial run but became used to relatively quickly.

It is also important to describe what type of data was collected during the observations and how this was done. Robson and McCartan (2016) describe several different approaches ranging from highly structured observations that utilise observation schemes, through to un-structured observations that are very open. I adopted what may be termed a semi-structured observation approach. I used a pre-determined framework, which involved taking detailed field notes alongside images of pupil work and teacher notes on the flipchart. The observation framework itself was guided by the data instruments section in my Theoretical Framework chapter, which helped me focus in on specifically how representations were being used. To do this, I used two iPads; one that was set up at the back of the room recording the lesson and one which I carried around during the lesson and used to write field notes and take photos, which I could then quickly add into my field notes document using a note taking application. These were then typed up and expanded and reflected upon using the video recordings as soon after the observation as was possible, to provide an accurate representation of what happened (Cohen, Manion and Morrison, 2007; Robson, 2011). In both instances this was a few hours later, on the same day as the observation. I was also able to go back to the video recordings during the analysis phase to compare my observation notes to the video itself and found this to be a highly useful strategy. Overall, I found that this approach enabled me to capture each lesson in a great deal of detail. During the analysis phase of my study this was particularly useful as I found that my memory of each lesson was very clear, which meant that as I was going through the data corpus, I could make links between the lessons and other sections of the data quite fluently. The framework I used is included in the Appendix chapter (appendix 2) alongside an example section of one of my hand-written lesson observations and the final typed-up version (appendix 3) so that the reader can analyse the way in which I used the framework.

4.6 Textbook Analysis

For each lesson observation conducted, I collected the teacher's planning materials and the textbook lessons used. Being able to access and analyse these materials allowed me to contextualise the use of representations in a broader way and look for the origin of any representations used. In particular, it was the analysis of the textbook that required careful thought and planning. Mathematics textbook analysis has a long history within education research however, often, it has been used as a tool to compare two or more different textbooks with the intention of shedding light about similarities and differences between opportunities on offer (Schmidt et al., 1997; Pepin and Haggarty; Charambolous et al., 2010). This study required a slightly different approach as the intention wasn't to compare one textbook to another, rather to better understand the phenomenon of teacher beliefs, knowledge, and use of representations, and the role of the textbook as part of this phenomenon. Therefore, including an analysis of the textbook in the research design was an important aspect of accounting for ontological depth as it provided an additional perspective of the phenomenon within my case study.

To approach the textbook analysis in a systematic way, it was clear that I needed to use a structured framework to help ensure that no important nuances were missed. Previous studies that have utilised textbook analysis have used methods designed to study very specific topics such as word problems (Santiago et al., 2022) or early number sense (Petersson, Sayer and Andrews, 2022), as well as usually being used to compare two or more textbooks (Schmidt et al., 1997; Pepin and Haggarty, 2001). Alternatively, other methods have been used to analyse changes in single mathematical texts over time (Morgan and Sfard, 2016). What became clear was that I needed to adopt a framework that I could adapt and use to meet the needs of my study. Charambolous and colleagues (2010) provide a useful overview of different approaches to textbook analysis and identify three different ways of approaching textbook analysis within the literature. They describe these as 'horizontal', 'vertical' and 'contextual' ways of analysing textbooks (ibid., 2010). Horizontal analysis refers to looking at the textbook as a whole, physical object and providing some description of it as an artefact (e.g., number of pages, chapters, size etc.). Vertical analysis is more in-depth and considers how a particular topic is treated. Contextual analysis considers how the textbook is used in a classroom context (ibid., 2010). The suggestion being that any one of the three methods might be used to analyse mathematics textbooks and that it is important for the methods chosen to meet the needs of any research being conducted (ibid., 2010). My study sought to better understand how teachers thought about fractions and their associated representations alongside studying their classroom practice in this area. As is outlined in the literature review chapter (section 2.4.2), the textbook can be theorised as being an active part of the meaning making process that is happening within the school maths classroom. Therefore, including the textbook within my approach to data collection was important because it highlighted one influential factor that may be important in answering the research question. Within my study, I decided to use the three methods outlined above as the starting point for my textbook analysis (Charambolous et al., 2010). Doing this enabled me to have a deep understanding of the structure, design and content of the textbook itself (horizontal and vertical analysis) whilst also understanding the way that the teacher was using it (contextual analysis). Essentially, utilising all three approaches helped provide data about the role of the textbook in the process of negotiating the meaning of fractions, which was important to answering the research question. The ensuing discussion will describe the approach taken to each of the three types of analysis, how I adapted them to meet the needs of my study, and how they supported the aims of my study.

The horizontal analysis is arguably the most straightforward aspect however, it is still important in providing some general information about the textbook itself for anyone not familiar with it already. Within this study I decided to analyse the textbook from the year group that the teacher in question was teaching as well as the textbook from the previous year as this would provide some potentially useful contextual information. I developed the idea of horizontal analysis by creating a framework that would prompt the analysis of a variety of general aspects of the textbook important to the context of my study (appendix 3). To develop the vertical analysis specifically for my study, I focused upon fractions and their treatment within the textbook from three different perspectives: types of representations used, fraction construct used, and progression of representations. Each of these perspectives were developed partly from the research by Charambolous and colleagues (2010), but predominantly from my theoretical framework (see chapter 3) so to provide more pertinent information to the research question. These three aspects of the vertical analysis were incorporated into the textbook analysis framework (appendix 3). The third element of textbook analysis concerns contextual use of the textbook by teachers. This includes how they interpret, amend, ignore, and apply the textbook content in the process of planning and delivering school maths lessons. Such contextual analysis is already accounted for within the whole research design as, within my case study, contextual data from lesson observations and semi-structured interviews were analysed alongside the textbook itself to provide this information.

4.7 Teacher Problem Tasks

To gather data on teacher's knowledge and beliefs about fractions, Kuntze's (2012: 275) framework has been used to devise five field-specific tasks that relate to the "content domain-specific", "related to particular content" and "related to a specific instructional situation" levels of globality. This approach is innovative in this field and makes my study different in design from previous studies that have sought to understand mathematical beliefs and knowledge. Data on the highest level of globality (generalized/global knowledge and beliefs) was collected through the stimulated recall semi-structured teacher interviews.

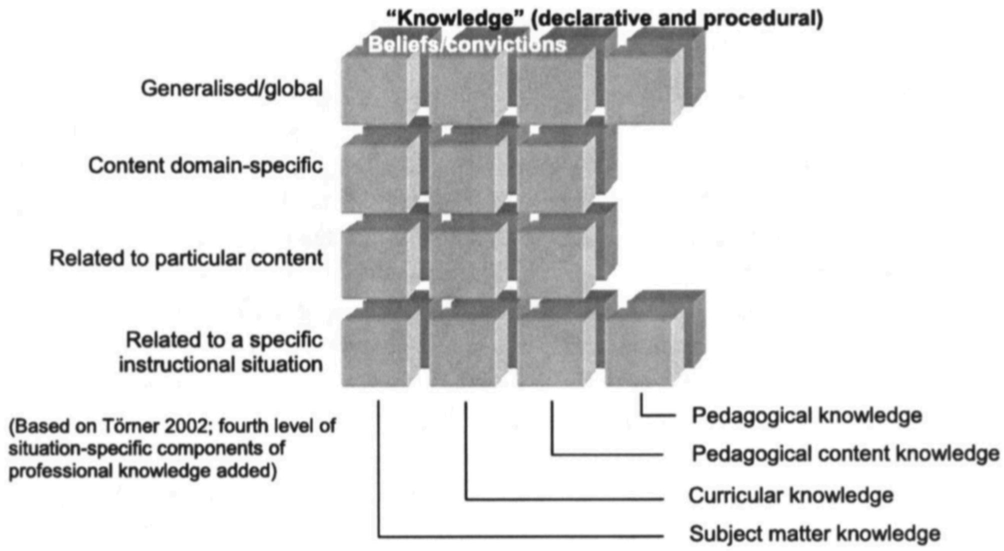


Figure 12 - Kuntze's (2012) theoretical model of teachers' professional knowledge

The tasks draw significant influence from Löfström and Pursianinen (2015) who used a similar approach in their study, aiming to use contextualised tasks to gather more specific data relating to teacher beliefs. Like Löfström and Pursianinen (2015), the tasks in this study were designed to be open-ended and not overly complex so that participants were able to discuss their thought process and actions whilst doing them. The aim was to use the tasks themselves as objects for discussion about mathematical beliefs and knowledge, therefore the discussion around the tasks was audio recorded and transcribed, as well as physical copies of the responses being collected. This method was designed to elicit more nuanced and context-situated information about beliefs compared to common self-report instruments such as questionnaires (Cobb, 2007; Löfström and Pursianinen, 2015). The aim was to get teachers to discuss their beliefs in context specific scenarios and to utilise metacognition and reflection as a way of understanding teachers' beliefs and knowledge (Löfström and Pursianinen, 2015). In this way, the design intends to acknowledge the situated and metacognitive elements of epistemological beliefs (Raymond, 1997; Hofer, 2001). Nevertheless, asking a teacher to carry out these tasks whilst discussing them and being audio recorded may well be considered quite challenging, and it was my professional relationship with the participant as an 'insider' researcher (Hellawell, 2006) that provided me with the pre-existing foundation that enabled me to collect this element of the data. I argue that had I presented this to a teacher with no pre-existing professional relationship, then the process would have been much harder and perhaps not a viable method. The five tasks are set out below along with some description of the rationale behind their design. Each task was trialled by mathematics teachers from both primary and secondary phases prior to their use in this study to ensure that they were fit for purpose and to avoid un-intended ambiguity.

4.7.1 Teacher Problem Task 1

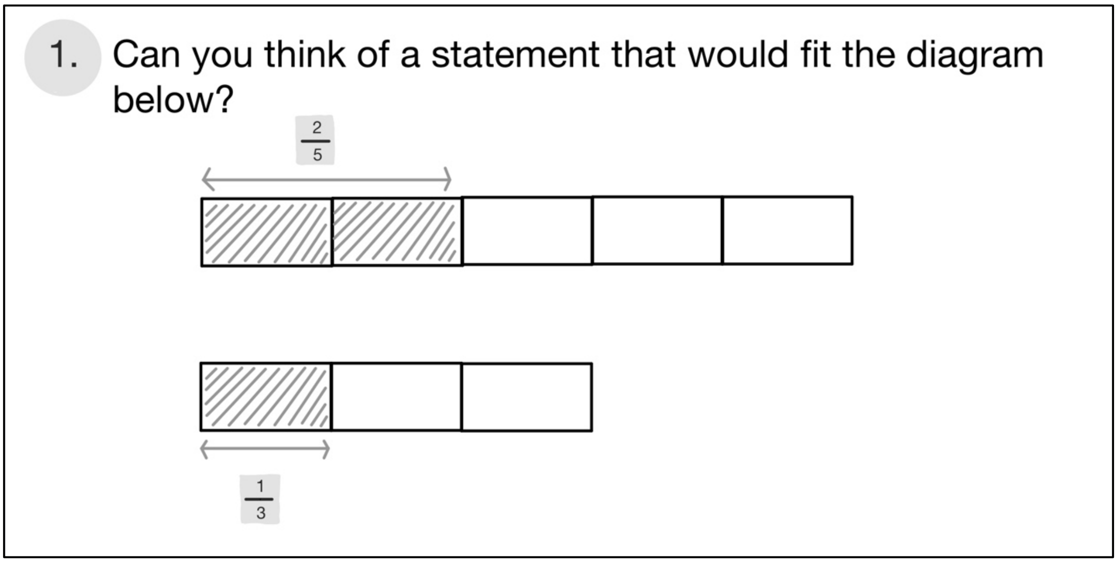


Figure 13 - Teacher Problem Task 1

This problem is designed to elicit teachers' epistemological beliefs and content knowledge relating to fractions. At first glance, the diagram is incorrect: it shows one-fifth being equal to one-third in size. The design here is to be able to see whether teachers can break from what may be considered a purely abstract use of fractions and create a context that enables the diagram to become mathematically correct. The ability to create a mathematically correct scenario to match the diagram should also elicit information about the participant's content knowledge: is their understanding of fractions such that they know you can have two fractions from different wholes and, to compare them, a different way of mathematical thinking is required (when compared to purely abstract manipulation of fractions)? An example of a statement that fits the diagram is: 'One person had £500 and they spent two fifths of their money. Another person had £300 and they spent one third of their money'.

4.7.2 Teacher Problem Task 2

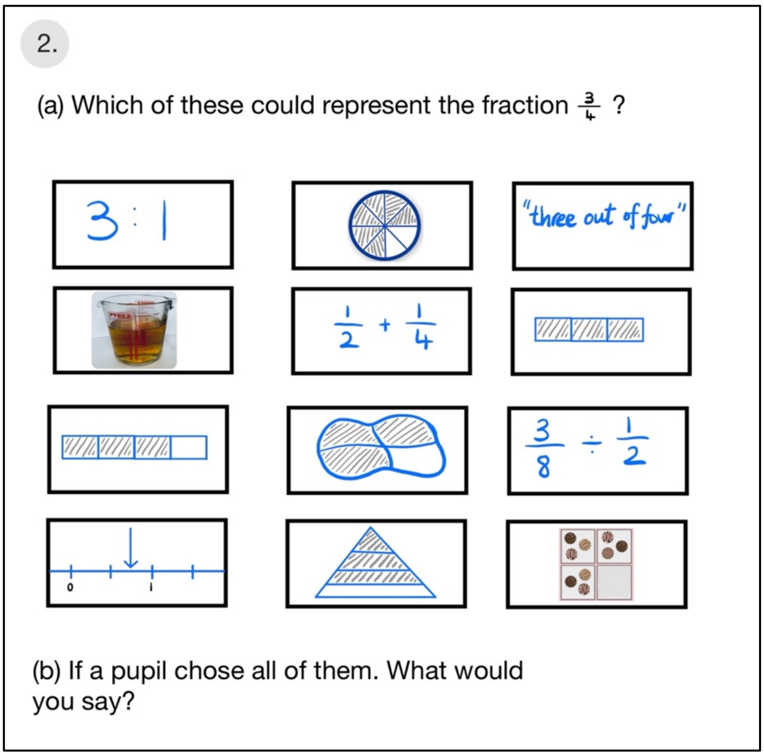


Figure 14 - Teacher Problem Task 2

This problem is designed to elicit participant's beliefs and knowledge about multiple representations and fractions. The wording of the problem is deliberately ambiguous: using the word 'could' allows for a degree of flexibility in response. For example, it is possible that the triangular diagram *could* represent three quarters; it is just not possible to say for sure without doing some very careful measuring. In fact, all the representations *could* show three quarters. A wide variety of possible options have been selected to ensure that most common representations are covered along with some that may be considered less common. Again, this problem is designed for participants to think out loud during the process so that data about their rationale for choosing can be obtained.

Part 'b' for this problem has been added to challenge participants to consider how they would respond in a more situated context (dialogue with a pupil). Thus, eliciting data about teachers' beliefs relating to multiple representations in the context of pupils' learning about fractions.

4.7.3 Teacher Problem Task 3

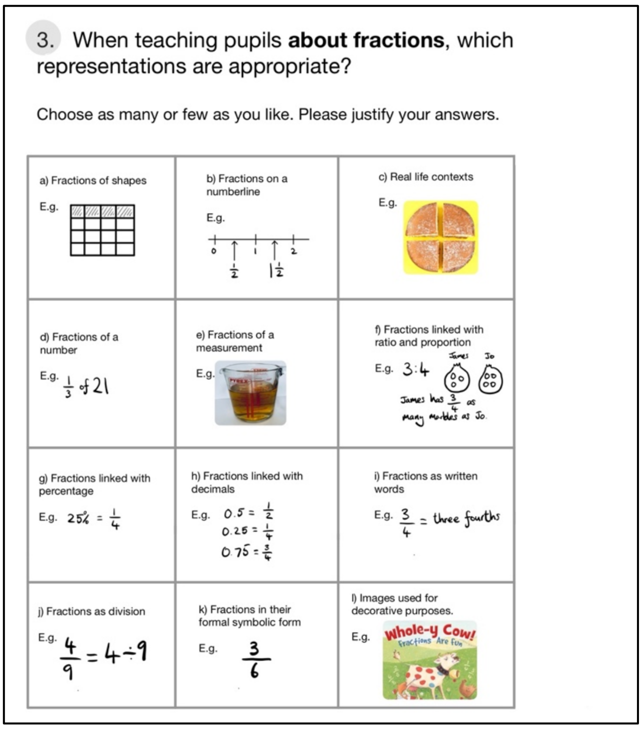


Figure 15 - Teacher Problem Task 3

Like problem 2, this problem is designed to identify participant's beliefs and knowledge of multiple representations relating to fractions. However, the key difference here is that it is asking them to consider which representations are appropriate for helping pupils learn about fractions. The aim is to provide further data about beliefs and knowledge relating specifically to the *teaching* of fractions and the way in which they believe representations should be used to support learning. A wide range of representations have been offered including one that is non-mathematical (representations for decorative purposes). These cover the core applications of fractions within mathematics as well as real-life representations of abstract ideas (e.g., the image of a cake is similar to a fraction of a circle). The main purpose of this task is not to test participant's knowledge of multiple representations but to get them talking about their own beliefs and knowledge related to these in the context of teaching fractions.

4.7.4 Teacher Problem Task 4

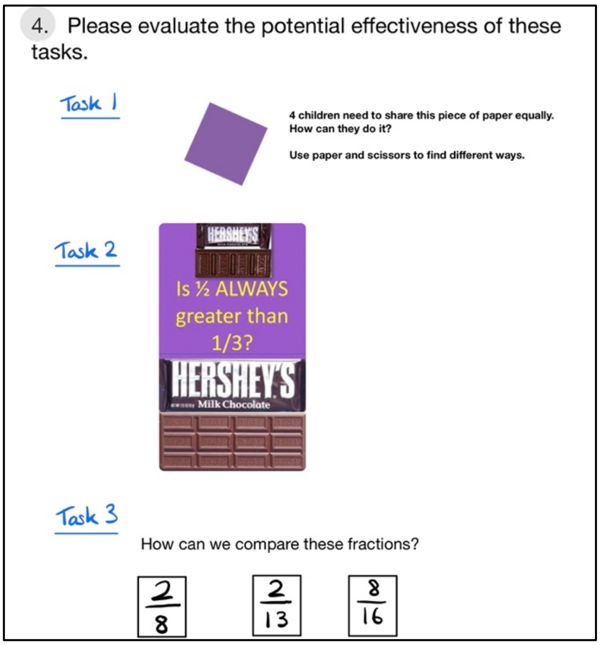


Figure 16 - Teacher Problem Task 4

Three possible tasks relating to fractions have been chosen and participants are asked to evaluate the potential effectiveness of them as teaching tasks. Again, this is left deliberately ambiguous because it aims to identify what teachers believe is an effective mathematical task for teaching fractions. This will require some extra prompts from the interviewer to ask about what they notice relating to the representations used. Task one has been chosen because of its prompt to use ‘real’ materials (paper and scissors). Task two has been chosen because it has a real-life image that is not particularly clear as to how it relates to the question and the question itself is also closed in nature. Task three has been chosen because it only has the formal symbolic representation used for fractions, yet it has a lot of potential for mathematical investigation if teachers were to choose to use multiple representations alongside it.

This problem should elicit data regarding teachers’ beliefs about using multiple representations in relation to fractions and specific tasks, in particular, what teachers believe about the reasons for using multiple representations (e.g., to engage pupils, to connect mathematics to real life, to deepen understanding of mathematical structures, or because they think they are supposed to).

4.7.5 Teacher Problem Task 5

5. Please watch the video clip and comment on the representations used in the lesson. https://www.ncetm.org.uk/resources/50319

Figure 17 - Teacher Problem Task 5

This video is on the National Centre for Excellence in the Teaching Mathematics (NCETM) website and is of a 'model' mastery lesson taught by a Chinese teacher. The focus is on fractions and watching/responding to this video should elicit data relating to teacher beliefs in the context of a specific instructional situation. The video is interesting because the teacher is direct in her instruction of the lesson yet appears to value and emphasise pupils' own representations relating to the mathematics being learned.

4.8 Data Analysis

Following on from the previous four sections that describe the data collection methods, it is now important to outline the way in which this data was treated during the analysis phase. Throughout this study, analysis took place concurrently with data collection and, although the process of analysis is described in formal terms below, it is important to highlight that the process was fluid and involved dialogic movement between the empirical data, and my own ideas and thoughts that were aided by the use of a reflective journal, alongside member checking with the research participant to ensure high levels of reflexivity (Atkinson, Coffey and Delamont, 2003; Doyle, 2007). A key consideration when approaching analysis was how to ensure that the data was treated in a manner aligning with my theoretical framework and critical realist stance. Taking this into account, I adopted a hybrid approach to thematic analysis which involved a combination of inductive, deductive, and retroductive reasoning. A point worth re-iterating here is that it was my decision to focus upon a single teacher that enabled me to conduct such an in-depth approach to analysis. In this section I will outline this approach and then go on to describe the details of this three-phase process, highlighting pertinent issues.

By way of introduction, thematic analysis (TA) is a popular analytical approach for identifying patterns of meaning within data (Braun and Clarke, 2006). Despite its popularity, in their seminal paper Braun and Clarke (2006: 77) described it as being “poorly demarcated”, citing that many use it without reference to philosophical underpinning and that it sometimes can fall prey to the “anything goes” critique of qualitative research. Since then, the reason behind this lack of clarity surrounding TA has been discussed by many and there are two key reasons for it, which are pertinent to this study. First, TA was often cited as a methodological approach to research, however it is now widely agreed that most of the research utilising TA, uses it as a *method* of analysis (Joffe, 2012). This does not mean that TA is atheoretical however, rather that it is flexible and different types of TA can be adopted to align with different methodological approaches (Terry et al., 2017). Second, it is somewhat misleading to think of TA as a singular method. Because of its flexibility, many different types of TA have emerged and become well defined, thus TA is more of an overarching idea with many different variations, each of which are a method in their own right (Braun, Clarke and Hayfield, 2019). For these two reasons, it is important that the type of TA used is clearly outlined, along with the way in which it aligns with the critical realist philosophical backdrop of this study.

In this study I adopted what has been termed a “hybrid approach” to TA, which has been used within constructivist research by others (Fereday and Muir-Cochrane, 2006: 82). This hybrid approach combines both inductive and deductive reasoning and seems to sit within what Braun and Clarke (2019: 594) call “codebook TA”, which they describe as being somewhere in-between interpretivist and positivist thematic analysis. The rationale for this choice was that it aligns with a critical realist stance in two important ways. First, it does not reside firmly within either the constructivist or positivist stance and therefore aligns with both critical realism and social realism which also aim to avoid doing this. Second, by using a range of different approaches to reasoning with the data, it is more likely that ontological depth will be represented within the analysis, because multiple perspectives of the data are offered through the different phases of analysis. Nevertheless, taking influence from critical realism specifically, in this study I also enhanced the concept of the hybrid approach by adding the idea of *retroduction* into the process. In practical terms, what this meant was that the theoretical framework was developed specifically so that it would provide the tools to go beyond the empirically observable and facilitate analysis that identified organising principles of behaviour. The choice of LCT within the theoretical framework is important here as this is a key aspect of the rationale behind the development of LCT by Maton (2014). As an explanatory theory it aims to be able to provide researchers with the tools to get under the surface of data and draw out these organising principles, hence supporting the retroductive reasoning process. The exact approach to hybrid TA used in this study was significantly influenced by that of Fereday and Muir-Cochrane (2006) who took a phased approach, first conducting thematic analysis using inductive reasoning, followed by the application of a theoretical framework using deductive reasoning. Additionally, influence was taken from Crinson (2007) and Fletcher (2017) who both outline the way in which a critical realist philosophy can be used within an approach to data analysis, with particular emphasis on the retroductive stage. Within my study I used a three phased approach and, although presented in a linear format here, it was an iterative process that involved moving back and forth between each phase of analysis, in line with much qualitative research (Terry et al., 2017). Nevertheless, as Bernstein (2000) recommends, my analysis did start with trying to ignore my theoretical framework and immerse myself in the data, conducting the reflexive TA phase first so as not to suppress any data (Braun and Clarke, 2019), in line with other critical realist research (Crinson, 2007). The three phases of analysis used were:

> *Phase 1:* Thematic analysis of the data using inductive reasoning
> *Phase 2:* Applying the literature-based data instruments element of the theoretical framework (section 3.3), using deductive reasoning
> *Phase 3:* Applying the ‘Specialization’ and ‘Semantics’ elements of the LCT toolkit (section 3.6), using a process of retroductive reasoning

Phase 1 involved conducting a thematic analysis using inductive reasoning to identify themes within the data. I took influence from Braun and Clarke (2006: 87) during this phase, applying their six stages of thematic analysis:

1. Data Familiarisation
2. Generating initial codes
3. Searching for themes
4. Reviewing themes
5. Defining and naming themes
6. Producing an initial report on these themes

The following is an account of these six stages. The process began with immersion in the data in a variety of ways. First, as I was the only researcher, I was there for all data collection and therefore experienced it first-hand. This meant that I immediately had a sense of the wider context of all the data. Additionally, I had also visited the school twice in the year before my study began, as part of my professional role, and this meant that I had a good sense of the school where the teacher worked and its contextual factors. Second to this first-hand experiencing of the data collection, I also made the decision to transcribe the audio recording of the teacher problem tasks as this would enable me to become more familiar with the data and match up any comments with the annotated sheets of paper that I had collected. The other two interviews were transcribed externally by a specialist audio-typist, however I found myself deliberately listening to the audio recordings of these interviews during this phase of analysis to get a better sense of the data so that I could contextualise the transcripts more accurately and add any missing nuances and correct errors. Doing this also helped me to mentally put the written transcripts back into the context of the interview, remembering more nuanced aspects of the interview experience thus enabling a more detailed and contextualised understanding of the data. Similarly, with my lesson observation field notes I found myself continually re-visiting the actual video recordings of the lesson to clarify any notes I had made. Through this initial process I became very familiar with the raw data which made all future stages of analysis more fluent. Once I was familiar with the data in this way, I then proceeded to look for any patterns so I could generate codes. This took place once most data collection had occurred, after approximately one month of data collection and familiarisation. These codes were largely “data-driven”, meaning that, at this point, I was trying to look for any recurring patterns within the data without imposing my own criteria (Braun and Clarke, 2006: 88). This was done by hand at first, using highlighter pens and sticky notes and involved looking at every transcript along with the lesson observation notes. After doing this, I then opted to utilise a Computer Assisted Qualitative Data Analysis (CAQDAS) programme and went through the process of re-coding the data again. I was then able to compare my handwritten coding with the version I had developed on the CAQDAS programme and refine all codes so that they accurately reflected the data, but also ensuring they were distinct. These codes were then grouped into initial themes, which were then reviewed and revised before generating final themes, following Braun and Clarke’s (2006) third, fourth and fifth steps. There are two important points to raise here. First, in this process I treated myself as *generating* these themes rather than them ‘emerging’ or being ‘discovered’ (Ho, Chiang and Leung, 2017). This is important because, as the researcher, I am not without my own philosophical stance and this means that in analysing the data and writing about themes, I am using my pre-existing knowledge and beliefs to manipulate the data into a theme – it becomes something I have generated (ibid., 2017). Nevertheless, because this phase adopted inductive reasoning, the intention with these themes was still for them to be data driven and therefore strongly linked to the data corpus itself (Nowell et al., 2017). Second, in combining the different codes into themes, I was looking for each theme to be an explanatory “central organising concept” (Braun and Clarke, 2019: 593) that referred to a specific pattern of meaning within the data corpus (Joffe, 2012). Nevertheless, there also needed to be coherence across all the themes so that they all related to one another, as well as the focus of this study. This is what Braun and Clarke (2006: 91) refer to as “internal homogeneity and external heterogeneity”. In practicality this meant that each theme needed to be clearly distinct from others, but also linked to them to maintain coherence across the analysis. This was not an easy process and there was a significant amount of re-labelling and double checking of what I had done before final themes could be defined. Some themes were collapsed and merged with others and other themes became apparent to me later in the process. In the spirit of academic rigour and transparency, I have included a description in the appendices to illustrate the process of initial coding and then grouping of codes into one theme (appendix 4). Once these steps were completed, a written report about these themes was produced before embarking on phase 2 of the data analysis process. Nevertheless, it is important to re-iterate here that, during the next two phases, I also kept coming back to this phase of analysis and making small changes. In this way, each phase of data analysis influenced the others.

Phase 2 of analysis involved applying the literature-based data instruments element of the theoretical framework (section 3.3), using deductive reasoning. In the words of Maton and Chen (2016: 30), I was using a framework devised from the literature as a “data instrument”, which could help guide my analysis by asking specific questions of the data, which are based upon the literature. Practically speaking, this meant that I used pertinent information from the literature to help guide my analysis towards important themes, as identified by previous research and by the research questions of this study. Throughout this process, I was constantly referring to the reflexive analysis and using the themes generated in phase one to support the application of the data instruments.

Due to the focus of this study, the data instruments were designed to draw attention to two key areas – ‘effective use of representations’ and ‘availing beliefs and knowledge’. The content of these instruments is outlined in the Theoretical Framework chapter (section 3.3). In conducting this phase of analysis, I systematically applied the two data instruments across the whole data corpus, using deductive reasoning to look for pertinent pieces of data. I found that there was a significant amount of overlap between the themes generated from the reflexive analysis (phase 1 of analysis) and this phase, however the application of the data instruments proved to be useful as it identified some specific ways in which participant was using representations, as well as helping explain something about her belief and knowledge system regarding school maths.

Phase 3 of analysis involved using a process of retroduction by applying two dimensions of LCT to the data; Specialization and Semantics, as previously discussed in the Theoretical Framework chapter (section 3.6). This phase of analysis aimed to explain the underlying structuring principles of the participant's beliefs, knowledge, and practice so that this could be used to help generalise the findings and contribute to the development of a working hypothesis. However, to be able to use LCT effectively, it was important to consider how the two dimensions were applicable to the context of my study, given that LCT is designed to be used across many different social contexts (Maton, 2014). In Bernsteinian terms, what was needed was a strong "external language of description" (Bernstein, 2000: 132), which would help maintain a clear relationship between the theory and the data from my study. For example, in Jackson (2016: 543), a 'knower' code within the specialization dimension is characterised in terms of literacy teaching where legitimate actions require learners to both have in-depth knowledge of texts, whilst developing a particular "gaze" through which they interpret these. In the same way, I needed to analyse the data to work out what constituted different codes within each dimension in relation to the participant's beliefs, knowledge, and practice. To do this, for each dimension I focused on aspects of the data corpus that were most relevant and split these up into segments. Each segment was then analysed using the relevant LCT dimension, attempting to develop a dialectical relationship between the data and the theory (Morais, 2002). Examples of each segment that represent the different codes within each dimension were then selected to act as a "translation device" (Maton and Chen, 2016: 43), which offers a way of moving between the data and theory, exemplifying what each code looks like in relation to the empirical data generated in this study. These are presented within the Findings chapter (section 5.7). For the analysis using the specialization dimension, the primary data came from the interviews, with some supporting data from the lesson observation notes and the textbook analysis. This data was grouped into segments and then analysed using the specialization dimension. For the analysis using the sematic dimension, the primary data source was the lesson observation notes, with some supporting data from the textbook analysis. Additionally, both lessons were analysed using the ‘semantic wave’, in a similar way to what Matruglio, Maton and Martin (2013) did with a history lesson. This involved splitting each lesson up into segments and then tracing the way in which both sematic gravity and semantic density changed in a temporal way throughout the duration of the lesson.

4.9 Ethical Considerations

Within my study, I take the stance that ethical decisions are not something that happen at the start of the research but are an on-going process involving the researcher continually asking questions of whether the study is maintaining an ethical position in relation to all involved. In this way, I do not treat ethics as being just a straightforward matter of operational practicalities (Wiles et al., 2005). As a bare minimum this meant that I needed to ensure that I had ethical clearance from the university ethics committee and ensure that the BERA (2018) ethical guidelines were met. Nevertheless, simply doing this can lead to ethics becoming a “one-shot” attempt at securing consent, which then gets forgotten about as the study progresses (Smythe and Murray, 2000: 320). I felt strongly that I needed to go beyond this and consider the potential risks to all involved and how these might be mitigated throughout the whole research process to ensure that my end product is ethically plausible. Therefore, there are two ethical issues that are important to outline here: the reputational risk to the teacher and their school, and the risks to the pupils involved. Additionally, for clarity, in this section I will also outline the measures that were taken to ensure data protection, which is particularly important given the personal nature of some of the data collected, such as classroom video recordings.

4.9.1 The Reputational Risk for the Teacher and Their School

Because my study focuses upon one teacher, there is a significant spotlight on them: their beliefs, knowledge and practices are detailed in great depth. This poses a potential reputational threat to them individually but also to their school as it exposes aspects of teaching practice that are specific to the participant, but also that are part of the school's mathematics policy. This is important as English schools operate within a high-accountability system where schools are scrutinized externally by a government funded agency and teachers are scrutinized and critiqued regularly as part of performance management systems, which often relate to pay awards (West, Mattei and Roberts, 2011; DfE, 2013b). Therefore, it was important to initially gain informed consent at the outset from both the participant and their headteacher. However, it was also important to maintain an ongoing dialogue about the research and the potential risks so that I did not slip into a mindset of assent as opposed to consent. In practice, what this meant was that I needed to always be open with the participant about the research and my interpretation of the data. As has been discussed previously (section 4.2.1), member checking and reflexivity were important aspects of this process and ensured that the participant was aware of the type of findings that would be outlined in the final writing up of my study and the right to withdraw any of the data was made clear throughout. To mitigate the potential reputational threat, I also made sure that all data was anonymised, and pseudonyms given where appropriate. However, it is perceivable that using excerpts from teacher interviews or examples of planning within any written findings will lead to other members of staff, or other schools, being able to identify the individual teacher or school from what they have said. Therefore, it was important to consider this when writing up findings and ensure that the participant was aware of this potentiality. Additionally, I needed to be aware of my position as an 'insider' researcher (Hellawell, 2006), and how this might affect their consent. I have a leadership role within a regional mathematics organisation (a 'Maths Hub') which gives strength to my research as I have considerable insider knowledge, however teachers could potentially feel coerced into taking part unbeknownst to me (although I do not have any official authority over the teacher who participated, it is important to recognise that this was still a possibility). Therefore, I needed to take extra care in making it clear that participation in the research was completely optional throughout the process, and that there was the right to withdraw elements of the data, or completely, from the study. In practice, my pre-existing relationship with the participant proved to be beneficial as we already had a professional relationship based upon mutual trust and this helped ensure that the process was ethically sound throughout.

4.9.2 The Risks to Pupils

Although the risks to pupils were minimal because they are not the focus of my study, they were still captured during video recording of lessons and therefore it was important to consider ethical issues surrounding this. The presence of a video recorder within lessons may well put some pupils in a position of unease so it was important to make sure that pupils consented to being in the video recorded lessons. Prior to any video recording, a letter giving information about the study with an ‘opt-out’ clause was sent to parents of pupils in classes where teaching was being video recorded. Before lesson observations and video recordings, what was happening was explained to pupils by their teacher and they were given permission to opt-out if they so wished. They were able to choose to sit outside of the camera shot or, alternatively, they were given an opportunity to be taught in another classroom for the research lessons. The option to not be in the video recording was given to all pupils. At the end of the lesson, pupils were reminded that they have the right to withdraw some or all of the data relating to them by asking the teacher. If there had been a case, for example, of extreme misbehaviour that was not pertinent to the study, then this would have been withdrawn from the data automatically. Once data had been gathered, it was also important to consider how this would be stored and how anonymity would be maintained.

4.9.3 Data Protection Measures

In recruiting a participant for this study, a central database of e-mails held by a local government funded organisation (the ‘Maths Hub’) was used to contact primary school teachers. These e-mails are held in accordance with English

General Data Protection Regulations (GDPR) and the study itself was deemed to be of interest to teachers whose details are held as a part of this database. An initial e-mail was sent out inviting teachers to take part in the study and once teachers replied, a specific case was chosen based upon the participant criteria outlined previously (section 4.3). Following this, an information sheet about the study was sent electronically and a short phone conversation ensued to discuss the research and any practicalities. At all times, the rationale and reasons for the research were made clear to the participant. Once data collection had started, all data was stored on an encrypted laptop and will be destroyed a year after the study is finished.

5 Findings

As is suggested within the literature (Gillham, 2000), the findings for this study were first written as an extended case study record. This included a broad range of details, from the basic description of what was done and when, through to a highly detailed writing up of each phase of analysis. In doing this, I took influence from Maton and Chen (2016) who recommend writing out findings in great length, including many data excerpts even to the point where it becomes unwieldy, before distilling it into a format that is more accessible to the reader.

This chapter will first present the sources of data used to form the case study, as a reminder to the reader, and will then outline some important contextual information about the teacher, Gillian, who is the case for this study. Although aspects of the textbook analysis are presented in a distinct section, I decided that also threading it throughout each of the other sections, referring to it where pertinent, was a more effective way to present this aspect of the findings. Additionally, I decided that providing a descriptive overview of each video recorded lesson would be useful, to enable the reader to better contextualise each phase of analysis. Therefore, the data analysis is presented in five main sections:

Section 1 - A description of the lessons observed
Section 2 – Key findings from the textbook analysis
Section 3 - Thematic analysis of the data using inductive reasoning
Section 4 - Analysis of the data using the data instruments, outlined in the Theoretical Framework chapter
Section 5 - Retroductive analysis using the 'Specialization' and 'Semantics' elements of the Legitimation Code Theory (LCT) toolkit, outlined in the Theoretical Framework chapter

5.1 Sources of Data

As is outlined in the methodology (chapter 4), data was collected using semi-structured interviews with stimulated recall, lesson observations, teacher problem tasks and textbook and document analysis. During the data collection, I visited the school four times in total. The first visit was to get to know the teacher and class better; to conduct the process of "habituation" (Brown and Dowling, 1998: 47). This included pretending to have the video recorder set up during the lesson (no actual filming was taking place). Following on from this, the data was collected in this order:

1. Interview 1 - Teacher belief and knowledge tasks (conducted on 28th October 2019)
2. Lesson observation 1 (conducted on 13th November 2019)
3. Interview 2 – stimulated video recall based on lesson observation 1 (conducted on 13th November 2019)
4. Lesson observation 2 (conducted on 28th November 2019)
5. Interview 3 – stimulated video recall based on lesson observation 2 (conducted on 28th November 2019)
6. Participant checking of early data analysis via e-mail and phone (conducted in early December 2019 across several days)
7. Interview 4 - Follow up interview questions via an on-line conference call (conducted on 12thth December 2019)
8. Textbook analysis (conducted across a few months in the summer of 2020)

5.2 Contextual Information About 'Gillian'

As is discussed within the Methodology chapter when explaining my case study approach (section 4.3), the context of the case being studied is important information for the reader, so that any interpretation of the data can be considered in relation to its specific context. This section serves to provide the reader with such information.

The sole participant of this study, Gillian, is a deputy head teacher at a small primary school based in the north-west of England. Her school has lower than average numbers of disadvantaged pupils. Gillian has been teaching for 16 years and specialised in mathematics during her teacher training. In real terms, this meant that she had an additional module in her final year of teacher training focusing on primary mathematics teaching, when compared to those who did not specialise in mathematics on the same course. Within the school, she is the mathematics subject leader and is also part of a national programme of teachers training to be ‘teaching for mastery specialists’ with a national government agency. Gillian and her school had been developing teaching for mastery as an approach, including the use of a textbook, for four years at the time of data collection. Practically speaking, this means that all teachers within the school (including Gillian) had attended three full days of external training from a local school-based organisation about how to use a textbook effectively, alongside some classroom coaching for all teachers where external specialists came in to work alongside teachers to support them with this in real-time during a school maths lesson. This was a similar experience to over 200 schools in the region at the time, who all also had the same professional development input whilst adopting the use of a school maths textbook and ‘teaching for mastery’. Since the data collection within this study, the number of schools experiencing this has increased considerably (at the time of writing, over 50% of English primary schools have engaged in this sort of professional development, NCETM, 2023a). At the time of data collection, Gillian herself had also attended one and a half days of training from a national agency as part of her ‘mastery specialist’ role, funded by the government, about the principles of ‘teaching for mastery’, which includes a focus on using different representations. This aspect of Gillian’s prior experience is arguably less common when compared to most English primary teachers as only a small number of teachers across England are selected for this role and access this training. However, Gillian was only into her first term in this new role and had limited experience of the accompanying training at the time of data collection.

During the data collection, Gillian was teaching a year 6 class which had pupils between the ages of 10 and 11 years old. Within her class, there were 17 pupils and only four of these were female. There were also 5 pupils within the class who the teacher referred to as having 'special educational needs', meaning that they each had some form of learning difficulty. The class had prior attainment from year 2 (6 and 7 years of age) that was in line with the English national average for mathematics. This would mean that the class would normally be considered as having an 'average' level of mathematics prior attainment, as judged by national tests. The classroom was set out with tables where pupils were sitting in groups of 2, 4 or 6. The teacher had a flipchart, maths display board and a projector, which were used together during the observed lessons.

5.3 Section 1 - Narrative Description of Observed Lessons

A strength of this study is that it is grounded in classroom teaching as opposed to proxies or reflections on practice alone. Therefore, the following section provides an important insight for the reader into Gillian's teaching. They are an integral part of the findings chapter as they demonstrate how the lesson segments used within the final phase of analysis, applying the LCT dimensions (section 4.8), fit together as a whole lesson. These lesson segments are reflective of natural breaks in the lesson where the teacher or pupils moved on regarding mathematical content, or type of activity. They have been generated from my lesson observation field notes along with the video recordings of each lesson. For further detail about each lesson, a more comprehensive lesson account, including images, has been included in the appendices (Appendix 6). These have been developed from my original field notes and serve to provide a full description of my interpretation of each lesson.

5.3.1 Lesson 1

About the lesson:

The first lesson that I observed Gillian teaching was focused upon simplifying fractions into their simplest form using common denominators. It is their second fractions lesson of the year and the first lesson, which they had the day before, was also focused upon the same thing. Within the textbook pages for this lesson, a mixture of rectangular area models, mathematical symbols and written words are used to represent fractions.

Table 10 - The first observed lesson broken down into segments

Lesson Segment	Description	Time
1. Beginning the lesson	Gillian begins the lesson by showing an image from the textbook on the screen (of a jam roll split into 12 equal parts). She provides pupils with white strips of paper and asks them to imagine that it represents the jam roll and to fold their paper in the same way the jam roll has been split up into 12 equal parts. Pupils take quite a lot of time doing this and discuss what they are doing amongst one another.	10 mins
2. Introducing the first problem	She then reveals further text that relates to the jam roll image, which talks about a character taking 8 pieces. A pupil comments that this is $\frac{8}{12}$ and Gillian writes this on the board. There is also a question asking whether he could have the same amount of roll with fewer pieces. Gillian asks the pupils to try and figure this out and they spend ten minutes doing this alongside drawing their own diagrams in notebooks and discussing their ideas about how to solve the problem. There is a lot of talk between pupils during this time and there are no quiet moments.	10 mins
3. Summary discussion of a solution using a rectangular area model	After pupils have had time to investigate the problem and discuss their ideas, Gillian interjects and leads whole class discussion about a way of solving the problem. The focus in this segment is on using rectangular area model representations to show that the $\frac{8}{12}$ could also be other fractions that are equivalent. The fractions are written as abstract symbols alongside the diagrams that Gillian is modelling. There is lots of pupil discussion amongst one another during this segment and pupils continue to draw diagrams in their jotter books.	8 mins
4. Summary discussion of a solution using abstract symbols	Gillian then moves the discussion onto another method that pupils had used – using purely symbolic manipulation to figure out different equivalent fractions. This is based upon the idea that they can multiply or divide the numerator and denominator by the same number to get equivalent fractions. Gillian spends most of this time encouraging pupils to see how this symbolic method connects to the diagrammatic one discussed previously. She uses phrases like “can you imagine…” to get them to visualise what the symbolic manipulation looks like with an area model. There is still a significant amount of pupil discussion.	8 mins

5. Lesson purpose summary	After going through the following two segments, where different ways of finding equivalent fractions were discussed, Gillian then asks the pupils what the purpose of the lesson is. She gets a variety of responses, and, through whole class discussion, they agree that the focus of the lesson is on finding equivalent fractions and simplifying fractions. During this, Gillian refers to the different representations she has modelled on the board along with the image from the textbook. This segment is predominantly teacher talk with some pupils interjecting.	4 mins
6. Pupil journals	She then asks pupils to write and draw about what they had been doing in their maths journals – these are a book they use in every lesson to record their thinking. Pupils spend about ten minutes doing this, during which there is a low murmur of discussion between pupils, but the room is generally quiet. In their journals, there are a variety of representations used including the area model, mathematical symbols and explanations written in English.	10 mins
7. Guided practice questions	In this segment Gillian asks the pupils to stop and then works through three 'guided practice' questions with them. With each question, she reads it with them out loud and then asks them to discuss it with partners and have a go at answering it using whiteboards, paper and notebooks as they see fit. These questions initially have area model representations given to go with them but the final one they do only has abstract symbols and the teacher encourages them to visualise area models to help them at this point.	12 mins
8. Lesson conclusion	The lesson concludes with Gillian briefly explaining what they had covered in the lesson and then telling them that they will practice simplifying fractions a little bit later that day. This means that Gillian was intending on providing time later in the day for pupils to complete some independent practice in their workbooks.	2 mins

5.3.2 Lesson 2

About the lesson:

The second lesson that I observed Gillian teaching was focused upon adding and subtracting fractions with different denominators. This lesson came two weeks after the first lesson described above. The lesson they were using from the textbook was the seventh fractions lesson (out of 17 in the fractions chapter). This means that she had covered five textbook lessons in 10 days, suggesting that Gillian had spent roughly two, hour long lessons for each lesson given in the textbook. Within the textbook lesson, the representations used are very similar to those in the first lesson observed, with the one exception being that pizzas are used as an initial problem context at the start of the lesson.

Table 11 - The second observed lesson broken down into segments

Lesson Segment	Description	Time
1. Introducing the representations	Gillian begins this lesson by showing an image from the textbook (of two pizzas and two children) along with an associated problem question, which is open ended in nature (it asks, 'What are some questions we could answer with this information?'). She then asks the pupils to represent the information by drawing their own diagrams. She allows approximately 7 minutes of quite loud pupil talk during which she roams the classroom looking at what they are doing. After this she draws attention to the fact that some pupils have drawn circles and others have drawn rectangles – she discusses the pros and cons of these, concluding that rectangles make it easier to manipulate the fractions and see the connections between them. She draws both representations on her flipchart. In a sense, here she is using her questions to elicit responses from pupils based upon her pre-planned agenda for the lesson.	8 mins
2. Pupils' own problem creation	After this, Gillian then asks the pupils to think about the different questions that could be asked with this information, and they start collaboratively working on this in pairs. Pupils use their jotters to draw a variety of diagrams and use some symbolic representations. Most pupils come up with problems involving the addition of fractions. There is a lot of loud pupil discussion.	7 mins
3. Pupils' own addition problems discussion	Gillian interrupts the pupils, who are busy discussing their own problems, and asks them what they have come up with. The first two suggestions are about adding the two fractions of the pizzas shown and Gillian leads some whole class discussion about these briefly, referring to the rectangular area model alongside the symbolic equation as a way of representing the addition of the fractions.	4 mins
4. Pupils' own subtraction problems discussion and investigating	Gillian seems keen to move onto subtracting fractions and deliberately asks a pupil, who has come up with a subtraction-based problem, to share their idea. She uses this time to draw out specific mathematical aspects of subtracting fractions that she wants to focus the pupil's attention on. One of the issues that the pupils seem to find difficult is the subtracting of two subsequent fractions from a whole number (e.g. $2 - \frac{1}{3} - \frac{1}{2}$) and Gillian spends quite a lot of time discussing this and using the circular and rectangular diagrams to 'see' what is happening. Pupils are quiet during a lot of this and Gillian's tone is more explanatory than in previous segments.	16 mins
5. Reflecting and summarising	Moving on, Gillian then states that she thinks that the pupils need some time to reflect on what they have been doing. She slowly talks through the ways in which they have discussed the subtraction of two subsequent fractions from a whole number. During this, she refers to the circular, rectangular and symbolic representations. There is no mention of the original problem context (pizzas) at this point. There is little pupil talk at this time.	4 mins
6. Pupil journals	Gillian then asks the pupils to draw and write about what they have been doing in their journals. Within pupils journals there is a mixture of representations being used, however it is clear that some pupils are still confused by the subtraction and they only record addition problems in their journal. There is little talk at this time.	10 mins
7. Lesson conclusion	The lesson finished with Gillian commenting that it had been a difficult lesson and that they were going to go over it altogether in	6 mins

	the next lesson. She reminds them again what they had done by referring to the circular, rectangular and symbolic representations on her flipchart.	

5.4 Section 2 – Key Findings from the Textbook Analysis

As is outlined in the Methodology chapter, the approach to studying the textbook involved three methods, horizontal, vertical, and contextual analysis (section 4.6). In this section, I will draw upon the horizontal and vertical analysis. The contextual analysis has been integrated into the other sections within this chapter.

5.4.1 Horizontal Analysis: describing the textbook

The textbook series in question was called ‘Maths-No Problem!’. As a textbook series, they produce materials for pupils between the ages of five and eleven years old. For each year group there are two textbooks, designed to be used one after the other across an academic year. There are also associated workbooks for each textbook as well as online teacher guidance. Each textbook and workbook pair are split up into chapters that cover specific topics (such as ‘fractions’) and each chapter is split up into individual ‘lessons’. Each individual ‘lesson’ within the textbook and workbook is split up into different sections described below.

- ‘In Focus’ – this section usually consists of a single problem designed to introduce the lesson content.

- ‘Let’s Learn’ – this section usually consists of characters who present different model solutions to the problem outlined in the ‘explore’ section.

- 'Guided Practice' – this section usually presents other questions based on the topic of the lesson. Pupils are not expected to write on the textbook in this section.

- 'Workbooks' – within each workbook there are several pages of questions which are designed to match the lesson content outlined within the textbook. The number of pages per lesson varies between one page and four pages.

The whole textbook series has child-like characters which stay the same from year 1 all the way through to year 6. These characters all have names and appear in every lesson to some extent providing questions, ideas and prompts for thinking. The overall presentation of the textbook pages is quite minimal. Most of each page is white, and the questions and colour images are spaced out. The workbooks are printed black and white and are designed for pupils to write in directly, meaning that schools would need to buy new workbooks each academic year.

5.4.2 Vertical Analysis: Understanding the treatment of fractions

Within this section, the findings from the vertical analysis of the fractions content in the textbooks will be outlined (a detailed sample of this analysis can be found in appendix 7). The findings from this aspect of the textbook analysis are reported here in relation to three key areas: representations, fraction constructs, and language and dialogue.

Most tasks (53% of them) within the year 6 fractions chapter have two or more representations within them. Many of these have three or more (28% of them). This suggests that the textbook authors place a high value on getting pupils to translate between different representation registers. Despite this, the different types of representation used throughout are quite limited. They are almost always one of the following: standard mathematical symbols, rectangular area model, circular area model (used less than the rectangular model), context images that directly link to the maths content, and some images of fractions as quantities (these are also linked to the context). Importantly, even the context images tend to be of ‘real life’ things that are either circular or rectangular in nature (pizzas, chocolate bars, jam rolls etc.) and somewhat mirror the area model representations used in the same lesson. This suggests that the textbook authors have tried to limit the exposure to many different representations, instead favouring the development of translation between a core few representations, and transformation within these core few representation registers. In contrast, within the Workbook, most tasks only show standard mathematical symbols. Only a very small proportion of tasks within the Workbook (16% of them) use another representation other than mathematical symbols, and these are all rectangular area models. It is worth noting that the tasks with an area model are at the start of the chapter (lessons 1, 2 and 3) and then re-appear when the lessons move on to focusing on fractions as an operator mid-way through the chapter. This suggests that the textbook authors are using the representation in the Workbook as a scaffold for when new ways of thinking are introduced but also, that constant use of the representation is not necessary for when pupils have developed a deep understanding. The sequencing of representations within the fractions chapter seems to follow a pattern. For the first task in each lesson (the ‘In Focus’ section), there is a tendency to have on average three or more representations used. These often include a real-life context, although there are no images used that are not directly related to the maths. Within each lesson, the number of representations used in each task then reduces, usually with the last one or two in the workbook having only standard mathematical symbols.

Somewhat related to the treatment of representations in the textbook, the number of different fraction constructs used was also quite minimal. The fractions chapter in year 6 is heavily weighted towards the part-part-whole and operator constructs, with lessons focusing upon these taking up 81% of the lessons (the rest being related to fractions as a measure). Nevertheless, there are no opportunities provided for pupils to iterate fractions, which might be expected within the part-part-whole construct. This is similar within the year 5 book as well. It is only later in subsequent chapters where other fraction constructs are used. This suggests that the textbook authors are emphasising the part-part-whole and operator constructs as the most important in developing a firm foundation in understanding and manipulating fractions. This might be seen as a weakness within the book by some, or as a confident statement about curriculum prioritisation by others. Either way, it suggests that teachers using the textbook will need to have a certain level of knowledge regarding the different fraction sub-constructs to notice them, and make the most of opportunities to develop them, within other chapters.

Although written language may be seen as another representation register, it merits a separate discussion based on the importance that the textbook authors seem to place on it, with 95% of the tasks within the year 6 fractions chapter having a written element to them. This is even the case with questions you might expect to not require any language. For example, in one task, pupils are asked to add and subtract fractions and instead of simply presenting some equations (e.g. ¼ + ½ = …), the textbook states “Find the value of each.” (Maths-No Problem! textbook 6A, p. 123). This suggests that the textbook authors place importance upon the use and development of mathematical language alongside mathematical understanding. This is supported within the lessons later in the year 6 fractions chapter where the written language of fractions (e.g. “one tenth”) is used to build connections to decimals. In one sense, mathematical language is used as a representational register throughout the book. In addition to this, the use of the common textbook characters is an important aspect of the way the textbook authors promote language and dialogue. It is interesting to note that almost every textbook page (91% of the pages in the year 6 chapter) has these characters appearing offering hints, questions, and statements designed to prompt thinking. This again suggests that the textbook authors have a strong commitment towards language and mathematical dialogue as part of learning mathematics. The textbook seems to offer a view of learning mathematics as a dialogic process involving reasoning and debate, rather than simply offering the exposition of mathematical knowledge.

5.5 Section 3 - Thematic Analysis Using Inductive Reasoning

Through the first phase of thematic analysis, both physical paper copies of the data, alongside CAQDAS software, were used to conduct an inductive reasoning process (see section 4.8). This began with the initial coding of the data, firstly using highlighter pens and post-it notes and then, later, using a CAQDAS programme. Alongside this, I also engaged in shared coding of some raw data with a university-based colleague acting as a critical friend (Biggs and Tang, 2011). By conducting this process in two separate stages, I was able to get to know the data very well and compare, double check and re-label what I had done. This then led to a gradual process of generating possible themes, which I began to write about, and then, finally, the generation of actual themes. Although I was using Braun and Clarke's (2006) six steps for my approach, the process was a messy, non-linear one and I went back over my initial codes several times whilst grouping them into themes. Although there is some overlap between these, each one has its own distinctive aspects which merit inclusion as a specific theme rather than a sub-component of another one. To exemplify this, I have included a description of the process of initial coding and then grouping of codes into one theme within the appendices (Appendix 4). Overall, there were 18 different codes identified that were then grouped into six different themes. The six themes are:

- Mathematics for the people, by the people
- Learning school maths requires resilience and reasoning
- Balancing pupil autonomy with teacher control
- Teacher and textbook collaborating
- 'Conversations' and representations to understand mathematics
- Using Representations for Mathematical Thinking

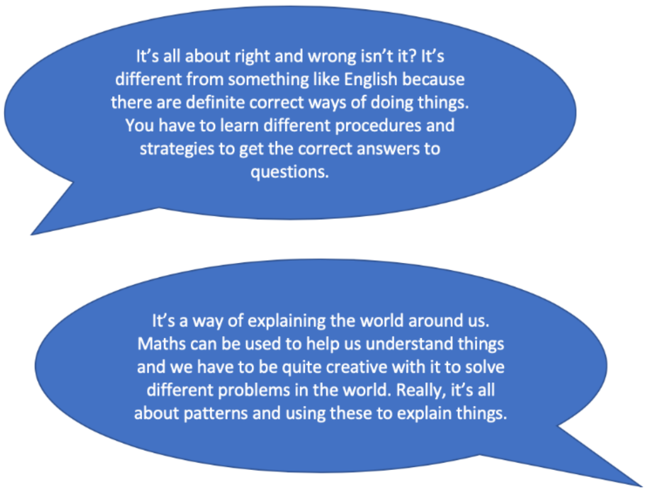
Each theme will be reported below by first providing a descriptive overview before moving on to discuss the nuances within that theme. Illustrative quotes are used throughout to exemplify themes, and these are representative of multiple quotes within the data corpus. Pertinent aspects of the textbook analysis will also be woven throughout each theme.

Mathematics for the people, by the people

A theme that was apparent across all the data was that Gillian had a firm belief in mathematics as a way of thinking that exists to be used by people. According to Gillian, mathematical activity should be social and purposeful activity, for people. This means that, for Gillian, pupils need to understand why they are learning something and what it is for. It also means that she believes the process through which a pupil develops mathematical meaning should be a social one, where communication of thinking is almost as important as the thinking itself. During the second interview, Gillian responded to the question, 'What is maths?':

> ***Interviewee****: That's a tricky question. What is Maths? Maths is – it's kind of – it's a way of understanding – in a way it's functional to life so for me Maths is what you need it to be so if you need Maths to be, to get you somewhere on time then that's what your Maths is I think. It's functional to what you need it to be. Does that make sense?*
>
> ***Interviewer****: Yeah.*
>
> ***Interviewee****: So the average person's Maths is very different to somebody whose job is more mathematical so engineering people, they need it to be more purposeful to them so I think, for me, Maths is what you need it to be.*
>
> [Interviewer and Gillian, Interview 2]

In this excerpt, Gillian emphasizes mathematics as a purposeful way of thinking. She displays a belief in it being a subject that is there to be used for the needs of people. In this quotation alone, she twice repeats her belief that maths '*is what you need it to be*' and refers to people with different types of jobs needing to use mathematics for different purposes. This suggests that she sees mathematics as a flexible subject that can be used in different ways to suit different social needs. Nevertheless, it does seem that she may be making a distinction between more specialist mathematics (she gives the example of engineering) and everyday mathematics, suggesting that different people will need mathematical skills and knowledge of varying degrees of complexity. This is not something that re-appears within the data, but it is a slight anomaly that is worth noting because it suggests that she may see school maths as having different purposes for different pupils. It could mean that, for her, some pupils need to learn basic everyday mathematics, whereas others need more complex mathematical understanding for future jobs they may have. Nevertheless, as previously mentioned, this is only one small aspect within the dataset and not a theme. In the same interview, one of the questions involved looking at two fictional quotes about mathematics and commenting upon them:



[Fictional Quotes from Interview 2]

Interviewee*: Yeah so that on the first one. I'd be more inclined to lean towards the second one, that it's understanding things and it can be quite creative and it is about looking for patterns. I'd lean towards that one but then I suppose there is an element of it where it is procedural.*

[Gillian, Interview 2]

In this excerpt she describes maths as a creative subject that involves looking for patterns and she aligns herself with the second quote, which emphasizes the socially constructed nature of mathematics, however she also acknowledges that there is an element of things being more '*procedural*' at times [Gillian, interview 2]. This suggests that she holds a belief in mathematics as a broad subject discipline that does involve procedural aspects where there are right and wrong answers, but that it is also much more than this. When asked about these more procedural aspects where there are right and wrong answers, she emphasized that it was a pupil's ability to reason, or justify, that was most important, commenting that, *'I think it depends on the reasons behind it [that] the child can give'* [Gillian, Interview 2]. This suggests that, even when it comes to the aspects of mathematics that involve definite right and wrong answers, for Gillian, it is the communication and justification of the mathematical thinking process that is most important. This supports her beliefs about mathematics as a social subject that is there for a purpose because, to her, it is the reasoning behind an answer that is most important, not necessarily the answer itself. The textbook analysis seemed to show the textbook as supporting this belief. In each lesson there are characters of young children who share their answers to problems with their reasoning mapped out for pupils to look through. These are presented as a sort of dialogue between the textbook characters and the reader. This suggests that the textbook scheme that Gillian was using was supportive of her belief about mathematics as a social subject where reasoning is of key importance.

Her beliefs about mathematics as a social and purposeful subject also extend to her beliefs about the way school maths should be taught. During the second interview, Gillian emphasised the importance of teaching so that pupils understood the purpose of what they are learning about and that contextualising the lesson content in real life scenarios was important *'because otherwise they see it as this thing that is never going to be used'* [Gillian, Interview 2]. This links with her beliefs about mathematics being '*what you need it to be*' [Gillian, Interview 2] and she is suggesting that, for her to be able to motivate pupils to learn the mathematics in the school curriculum, she needs them to see it as something purposeful for them. Without this, she describes the use of fractions as *'just a pointless exercise'* [Gillian, Interview 1]. Again, this re-iterates her belief in mathematics as a social and purposeful subject, not just something to be done for the sake of it. It is worth raising here that the textbook analysis showed that around one quarter (26%) of all individual tasks in the fractions chapter she was using were problems that involved a real-life context. Almost all of these were the first problem in the lesson. In each observed lesson, Gillian began with a real-life scenario that was given in the textbook and spent considerable lesson time on this lesson segment. Again, this suggests that the textbook is supporting Gillian's beliefs about mathematics and is part of her being able to enact these beliefs. Despite this, Gillian seems to acknowledge that the textbook real-life scenarios are a little contrived:

> *…that point that one of the children made last, a few weeks ago, 'I don't understand the point of fractions in my life, when will I need them?' Well actually you might need it because you might eat half a pizza and you might want to know what's left and so we just try and bring it back to them to make it a bit light-hearted for a few seconds…*
>
> [Gillian, Interview 3]

Here, she is giving the example of a pupil wondering about the point of fractions, and she seems to acknowledge that the idea of using fractions to calculate pizza amounts is slightly contrived, but that it helps the pupils get into the right sort of mindset for learning. This example shows how there may be some contention between Gillian's beliefs about mathematics as being purposeful for real life, and the content of the curriculum she is teaching. By their very nature, fractions (those taught by Gillian at upper Primary school level) have limited direct relevance to real-life and are important because of the stepping-stone they provide into the more complex mathematical thinking that is required for secondary school maths. Within the textbook, the analysis showed that almost all the real-life contexts were either rectangular or circular objects such as pizzas, cakes and rolls. This seems to be designed to lead pupils into using area model representations rather than providing meaningful real-life scenarios. This presents a challenge for Gillian who believes that mathematics needs to be directly purposeful. This suggests that, although the textbook scheme seems to support this aspect of Gillian's beliefs and knowledge in several ways, when specifically looking at the area of fractions, there is a slight tension.

Within the data, there was also one anomaly which seemed to contrast with this theme. During the teacher problem tasks interview, she picked up on my use of the word 'fourths' in one of the tasks, pointing out that *'you've done what I don't* *let my children do'* [Gillian, Interview 1]. She went on to add that she would *'tell them it killed a maths fairy'* and reflected that perhaps this was something that had come from the way she had been taught – *'my teachers always instilled in me that it's a quarter'* [Gillian, Interview 2]. This shows how Gillian may have certain aspects of mathematical knowledge that are very fixed for her (that it should be called a quarter, not a fourth) and she acknowledges that this seems to have come from her own experience as a learner of school maths. Her teachers taught her in that way, and so she has carried that into her practice. Here, she is downplaying mathematics as social and purposeful in favour of a piece of fixed knowledge without logical reason. Although this was the only piece of data that highlighted such a contrast, it is important because it shows how Gillian's beliefs about maths as a social, purposeful, and human activity may not always prevail and that other influences, such as her own historical experience as a learner of school maths, also play an important role. Nevertheless, it seems that Gillian does hold a strong belief in mathematics as a social subject that is there to be used by people, and this has connections to the beliefs she holds about pupils as learners of school maths.

5.5.1 Learning school maths requires resilience and reasoning

This theme highlights what Gillian believes to be important for pupils when learning school maths. Particularly, she believes that pupils need to develop resilience in the face of difficulty and that this is connected to their ability to reason and prove ideas. This includes a range of attributes, or learned habits, that influence how Gillian plans for, responds to, and assesses pupils in lessons. In many ways, she seems to hold these beliefs about what she wants her pupils to be like, over and above pupils getting answers correct. For her, it is more important that pupils can communicate their thought processes via verbal reasoning and use of representations. Also, she wants them to be able to learn from mistakes made and be able to make their own choices about how to solve problems, independently from the teacher. In many ways, her beliefs about pupils learning maths can be described as helping pupils become comfortable, being uncomfortable in her lessons. She wants them to be able to manage challenge and difficulty and to be able to work outside their comfort zone, without letting it get in the way of their learning.

> ***Interviewee**: …as a class, we spent a lot of time in Maths lessons thinking about 'right how do we feel when we're out of our comfort zone?' We feel uncomfortable and then we have this choice that we make so that's on our wall and so I'm very aware that they need to feel – it's OK to feel uncomfortable but we've got to push through that to get that pride at the end but I'm very aware that they do have to have that moment of settling down.*
>
> [Gillian, Interview 2]

In the quote above, Gillian is discussing the way she began one of her video-recorded lessons. In the lesson, she spent ten minutes at the start getting the pupils to think about some mathematical information related to fractions and how it might be represented by folding strips of paper. When asked about this, part of her rationale for the use of lesson time was related to this theme and the importance of pupils being ok working out of their '*comfort zone*'. She expresses a belief that pupils need to be able to cope out of their comfort zone and that it is her job as a teacher to help them develop this. Rather than try and avoid teaching lessons where pupils will struggle, she actively seeks out these experiences for her pupils and uses them to help develop what she believes to be important attributes. Related to this, it appears that she holds a belief in pupils developing greater levels of independence in their learning as she is aware, in her own words, '*there are times in life when I'm not going to be sat next to them telling them not to use that, to use this'* [Gillian, Interview 3]. This is an important aspect of Gillian's beliefs because it demonstrates her wider goals for pupils, in relation to pupil autonomy, rather than just successful learning of school maths. When she is teaching school maths, she is seeking to develop more than just mathematical knowledge within the pupils. Her aim is to help them develop into independent learners who can cope with difficulty. As I was beginning to generate this theme, I asked Gillian directly about it and, interrupting my unfinished question, she commented that *'It takes resilience'* [Gillian, Follow-up Interview] and went on to explain this further:

> ***Interviewee**: //They need to have that kind of mindset that, you know, 'I can have a go at this, I can do it". And then they have that logical thinking to work through it. Um… That's more important than anything because a gifted kid that can just do maths really quickly and doesn't have resilience struggles… So, for me it's resilience that's the most important thing. Completely… Erm… I suppose then they need… do they need? Or… Do they need to be given?*
>
> [Gillian, Follow-up Interview]

Gillian provides a relatively confident and certain response to the question about pupils as learners of school maths; so much so that she interrupted the question to provide her answer. It is interesting to note in her response that she contrasts resilience with mathematical ability without any prompting. She seems to believe that being a resilient learner is more important than ability in maths, suggesting that some pupils might be gifted, but if they are not resilient, they will still struggle in school maths. It is important to note however, that she also mentions that pupils, who are resilient, will also have '*that logical thinking to work through it*'. This suggests that it is not just resilience that she values highly, but also the ability to think logically and reason, and it is the combination of this alongside resilience, that leads to a successful learner of school maths.

> ***Interviewee**: Well, you know, in general we're always saying to them you know that so prove it because I don't want them to just say 'well I just know it'. But why do you know it? What is it that you know? And some of them were able to prove that by doing the common multiples so working in that abstract with those problems, but some of them were then proving it with a representation. They've probably got used to the fact that now, as part of our Maths lessons, they do have to prove it…*
>
> [Interviewer and Gillian, Interview 3]

This quote from the third interview references a key point that I had noticed in one of the observed lessons. Two pupils had been working together and, without any adult prompting, decided that they needed to '*prove*' their idea and by this they meant using a variety of drawn representations and verbal reasoning to explain what they had done. From the lesson observations, I had noticed that this was not a one-off occurrence but something that seemed ingrained within the culture of her classroom. Gillian affirms this and, again, contrasts this with just getting the answer right. For her, a successful learner of school maths does not just get correct answers, they can also provide a strong rationale for why they believe it is right. Related to this, the textbook analysis showed that almost every page of the fractions chapter she was using (91% of all pages in the chapter) presented images of characters offering ideas and reasons that were often connected to different representations. This suggests that Gillian's belief in the importance of being able to reason is also supported by the textbook resource she is using. Whilst discussing this, Gillian reflected on what her pupils were like four years previously, before they had been using the textbook. At first, they found the constant requirement to justify and reason *'really hard'* [Gillian, Interview 3] and demonstrated a reluctance to do so, but after four years of this as an expectation in lessons, it had become normal. This is something that Gillian strongly expresses as important throughout the data. She believes that pupils should be able to reason and prove their ideas and that this is partly linked to school maths being a creative subject, stating that *'you can get creative with how you're going to do it'* [Gillian, Interview 2].

She sees the habit of proving your ideas and finding different solutions, using multiple representations and verbal reasoning, as a more creative aspect of mathematics and one that is important for pupils to gain experience of. Not only does this show something about what Gillian values in a learner of school maths, it also seems to connect to her beliefs about maths as a social subject that is there to be used for a purpose. She wants pupils in her class to see school maths in the same way she does – as a purposeful and creative subject, and this is backed up by the way she teaches and the textbook resource she uses, placing important emphasis on proving and justifying ideas.

This belief in the importance of being able to reason and justify thinking also forms a major part of the way that Gillian differentiates between pupils in her lessons:

> ***Interviewee**: I think the more confident child or the more able mathematician is able to just do that problem and explore it in a million ways to represent it and to make those connections that it doesn't matter what I'm doing, I've got this deep learning and I can*

represent it as a bar, I can do this, I can show you this, I can apply it to this problem. However, those children who are a little bit struggling learners and it's a resilience problem and it's that decision of if you've got actually the efficient method, we have explored the maths behind it but you're not quite making those connections, whilst I know in terms of mastery and deep learning I should be pushing you to use them efficiently, if you've got the efficient method …

Interviewer*: It's what – so you're sort of saying you're making a judgement about what have I got time to do and what's the most important?*

Interviewee*: Yeah, which is sometimes part of the problem of being a Year 6 teacher. You've always got that in the back of your head.*

[Interviewer and Gillian, Interview 3]

In this excerpt Gillian explains what she believes a more '*able*' pupil is like. She strongly emphasizes that, for her, an able pupil is one who can use multiple representations to show different ways to solve a problem. This is something which is found throughout the data – Gillian believes that to be successful in school maths, it is important to be able to use multiple representations and verbal reasoning to be explain and prove what you know. The textbook analysis seemed to also support this, showing that 53% of the tasks within the fractions chapter utilised two or more different representations. Importantly, this is how Gillian seems to assess a pupil's ability in her maths lessons. She sees pupils who can do this, as pupils who are more advanced. Consequently, she sees pupils who struggle as those who are less confident in their use of representations and verbal reasoning. In the quote above she suggests that a struggling pupil is more likely to be reliant on using one type of representation or method. Again, this appears throughout the data, and she consistently talks about struggling learners as those who need more support to reason and use different representations, and advanced learners as those who can already do this well.

Nevertheless, importantly, in the above quotation, Gillian also refers to '*the problem of being a year 6 teacher*'. This reveals a tension in Gillian's belief system. Although Gillian believes it is of high importance that all pupils can use different representations and reason about these, she also is aware that there is a high-stakes national test at the end of year 6 and her school will be judged based on pupil outcomes (this is what she is referring to as a problem with being a year 6 teacher). Therefore, she seems to suggest that at times, this belief is put aside in favour of helping a struggling pupil develop the ability to use at least one efficient method or representation successfully. Despite this only appearing once throughout the whole dataset, it highlights an important aspect of Gillian's beliefs. She holds quite strong beliefs about the importance of resilience and pupils' ability to reason and prove ideas, however there are times when this is overshadowed, and it seems that ensuring pupils can at least get correct answers in a high-stakes national test might be one of them.

5.5.2 Balancing pupil autonomy with teacher control

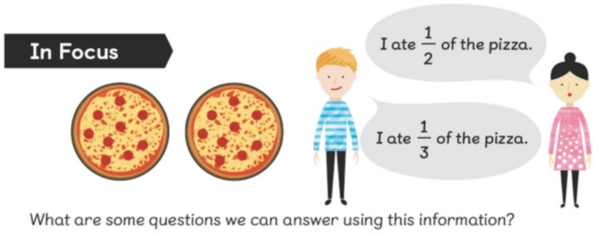
This theme is all about how Gillian attempts to tread a fine line between allowing pupils autonomy and control of their learning, whilst also maintaining quite tight control over the lesson content, direction, and outcomes. As a theme it is closely connected with her beliefs about pupils as learners, discussed previously, but also to her use of the textbook which is explored in a subsequent theme. Throughout the dataset, it is clear that Gillian thought it important to hand over some control to pupils in her lessons, primarily so that they developed into independent learners, however also that she highly valued teacher knowledge and felt that using this to direct pupils' learning towards specific intended outcomes in her lessons was important. It was apparent that there were some aspects of Gillian's practice where she was more than happy to take total control and make this clear to the pupils, and these tended to be aspects that were specifically focused on the mathematical content of lessons. However, there were also aspects where she was very informal and allowed pupils a great deal of control themselves and these tended to be more general aspects of practice such as behaviour during discussions and the use of whiteboards and notepads.

During both observed lessons (section 5.3), I noticed that during whole class discussions, Gillian did not require pupils to put their hands up when offering ideas, and that she allowed pupils to carry on their own conversations whilst she was talking. When asked about how this reflected her everyday practice, she commented *'yeah, that was a normal lesson'* [Gillian, Interview 2]. Gillian does not maintain very rigid control over pupil behaviour during class discussions. She seems happy to allow some pupils to carry on talking amongst themselves whilst she is talking and is also happy for pupils to shout out and do what she describes as '*heckle*' her as the teacher [Gillian, Interview 2]. This might seem as if this is a teacher not in control of the class however, this is something that Gillian is actively seeking. During the first stimulated recall interview, Gillian pointed out an interaction she had with a pupil:

> ***Interviewee****: … for some reason B's is on square paper like this and I asked 'why's yours square?' and she's like 'I dunno' …*
>
> [Gillian, Interview 2]

Here, she has noticed that a pupil is using a book with squared pages rather than blank pages as the rest of the class are using (using blank pages was the general school policy). To some this, alongside the pupil's response, might seem unacceptable and perhaps even rude, yet Gillian just accepts it and moves on, not seeming to be bothered by it at all. Although this is only one very small interaction, it is representative of the general way in which Gillian interacts with her class. She seems to be happy with an informal approach to behaviour management where there is very loose control from the teacher. Additionally, she also actively encourages pupils to have autonomy over their own learning by taking notes using jotters, which she describes as *'an informal book'* that she does not really look at or mark [Gillian, Interview 2]. She describes the allowance of this freedom for pupils to manage their own learning as *'massively'* important to her teaching, because it allows pupils to learn freely, without fear of judgement, because what they put in their jotter '*doesn't matter if it's right or wrong'* [Gillian, Interview 2]. These examples show how, with some aspects of her teaching practice, Gillian maintains low levels of control and is actively planning for pupils to have autonomy over what they are doing, closely linking this theme with her beliefs about pupils as learners of school maths.

Whilst the examples so far have demonstrated Gillian's loose and informal approach to elements of her practice, it was also found that she maintained quite tight control over other elements, which tended to relate more closely to the mathematical content. In one of the observed lessons, Gillian introduced the lesson with the problem seen below.



After about 7 mins of pupils doing their own problems, T gets them to stop and discuss different questions they have thought of.

[Field notes, Lesson Observation 2]

During the whole class discussion of this problem, despite a wide variety of pupil responses that were mainly focussed on adding the amounts of pizza eaten, Gillian drew the children's attention to how much pizza was left and spent by far the most time on this. Gillian explained how she '*needed*' [Gillian, Interview 3] them to see it as a subtraction question suggesting that, in this part of the lesson she is taking control as the teacher and explicitly directing them to think about the information in terms of subtracting fractions as opposed to the addition questions that they had come up with themselves. This is likely to be partly informed by the requirements of the formal national curriculum, but also the textbook content for the remainder of this lesson, which focuses on pupils being able to subtract two fractions from a total. In this instance she also goes one step further in the direction of teacher control. When a pupil gets it wrong by subtracting the two fractions from one pizza (rather than the two shown), the teacher tackles this head on and tells the pupil that they have got it wrong quite firmly. This demonstrates that, when the focus is primarily on the mathematical content that she wants the pupils to learn, Gillian maintains high levels of control. She still allows the informal approach to class discussion, with the pupil '*heckling*' her, however she goes on to tell the class that it is wrong and explains why. This is also representative of the wider dataset where she quite regularly discusses what she wants or needs pupils to see and how she goes about making this happen in her lessons. Where the focus is specifically on the mathematical content that Gillian is focusing upon in her lessons, she maintains quite tight control over the lesson, and this seems to be connected to her beliefs about the importance of teacher subject knowledge:

> ***Interviewee****: Implications for teaching Maths in school can also be teacher's subject knowledge, massively. That's a big – and it's not just the subject knowledge and the pedagogical knowledge. So for example knowing that that fractions lesson, knowing the common factors and multiples that are really engrained in that. If that hadn't have been taught first then the kids would struggle.*
>
> [Gillian, Interview 2]

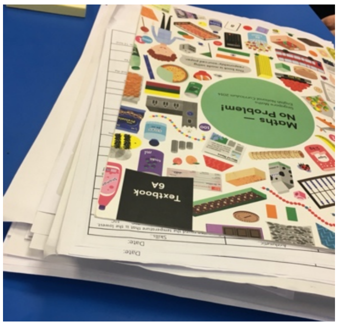
Here, Gillian is emphasizing her belief in the importance of teacher subject knowledge. She feels that it is important for teachers to have a good understanding of the mathematics they are teaching, and she related this to the needs of pupils who '*don't naturally make those links*' [Gillian, Interview 2], suggesting that her belief in the importance of subject knowledge is driven primarily by her need to help pupils who struggle. This begins to explain the tension that exists between her belief in pupils developing as autonomous learners, but also her maintaining quite tight control over the mathematics in her lessons. Perhaps one reason for this is that she sees some pupils as not being able to naturally make connections in mathematics and therefore directs them more explicitly at certain times. Nevertheless, it is not as straightforward as this and her beliefs in this area have multiple dimensions:

> ***Interviewee****: … Just because I'm standing there as the teacher does not mean that I'm the font of all knowledge and it doesn't mean that I can't get things wrong and sometimes with mistakes on purpose I use them as teaching points…*
>
> [Gillian, Interview 2]

On the one hand, Gillian highly values her own subject knowledge yet, on the other, she also understands and wants the pupils to understand, that she is not the '*font of all knowledge*'. It seems that Gillian is trying to navigate the fine line she has set herself between teaching for pupil autonomy and teacher control to ensure pupils learn specified content. Across most of the dataset, it seems that she manages to achieve this. Her lessons are typically full of opportunities for pupils to make decisions for themselves, nevertheless, most of the time, she is carefully managing the focus of their attention through her use of questioning, teacher modelling and explaining, so that her pre-planned mathematical focus in maintained. This is also closely connected to her use of the textbook as a tool which helps guide her through the mathematical content she is teaching.

5.5.3 Teacher and textbook collaborating

This theme is closely tied to the previous two themes discussed ('Learning school maths requires resilience and reasoning' and 'Balancing pupil autonomy with teacher control') and is also connected to the context of the overall study. Gillian's school uses one of the UK government-approved textbooks to support mathematics teaching across the school and the role of this textbook was prominent both in the lessons observed and the interviews. This theme demonstrates that she used the textbook as the core backbone of her maths lessons and placed great confidence in its content. However, she also holds her own subject knowledge confidently and utilises this to help her plan from and adapt the textbook, to suit the needs of the pupils in her class. For Gillian, the textbook provides an additional voice of mathematics teaching, contributing to her planning and delivery of lessons. In a way, her use of the textbook is collaborative in nature – she is collaborating with the textbook to create her maths lessons. Below is an image of her copy of the textbook, with her own notes inserted from previous years having used it, alongside an interview excerpt where we were discussing her use of the textbook.



[Field Notes, Lesson Observation 2]

Interviewer: *Yeah OK. Is there anything else you want to talk about, particularly about your planning? So basically what you're saying – rather than doing a written plan, you've got the textbook, you're reflecting on what they did in the previous lessons and what you know from previous years when you've taught those lessons? Is that right?*

Interviewee: *Yeah.*

Interviewer: *And then basically just thinking it through?*

Interviewee: *Yeah, I spend a bit of time thinking it through. I normally do it the night before based on the last [lesson] – I think right how did Maths go today? What's tomorrow's lesson? What's tricky about that? To be fair sometimes I've looked through, like I said earlier, in like my lunchtime straightaway afterwards. You've always got in your head what's coming up anyway…*

[Interviewer and Gillian, Interview 3]

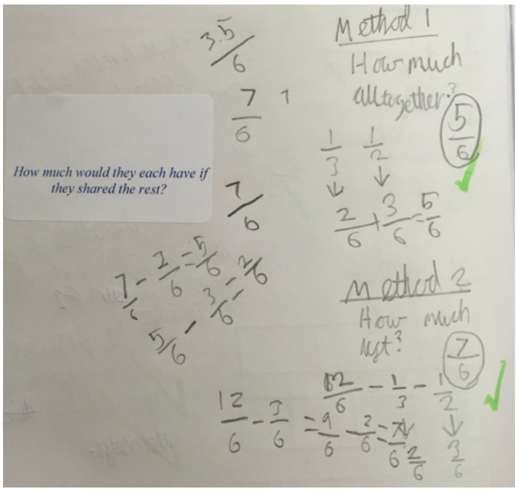
In this excerpt, Gillian is explaining how she plans for her lessons, and it seems as though the focus is upon the mathematics and how she can help pupils learn it rather than other common issues such as finding content for the lesson or making resources. This was observed in both lessons where most of the content used came from the textbook resource and each lesson was focused around solving the problems and using the representations given in the textbook. In both lessons there also appeared to be very few additional resources used by the teacher. In the first lesson the pupils had strips of paper and in the second one they were just using whiteboards and jotters. The textbook analysis showed that there were no instances of manipulatives being suggested by the textbook in this chapter, therefore any choice to do so came from Gillian. In both lessons the teacher had screenshots of the online textbook resource that were displayed on the screen alongside a flipchart, which she used for her own representations. There were no other materials that had been prepared by the teacher. These observations seem to support the idea that planning time was predominantly focused upon how she could help the pupils learn the mathematical content of each lesson. It is likely that the use of the textbook is partly what has enabled Gillian to spend her planning time in this way – the content has already been provided and this leaves her with more time to think about pupil learning and how she can enable this. This also connects closely with the previous theme which showed how Gillian maintains tight control over the mathematical content and focus of her lessons. It is important not to downplay the role of the textbook in this process and this is something that she clearly expressed in the interviews. When asked about her decisions about what representations to use in lessons her instant reply was that *'it [the textbook] told me to'* [Gillian, Interview 3]. Nevertheless, she did not blindly follow the textbook in this sense. During the second observed lesson, it had been clear that Gillian was wanting the pupils to use rectangular diagrams, not circular, and this was reinforced during the interview afterwards. However, the textbook initially presented the fractions as pizzas (using a circular model). The fact that she opted for the textbook approach to begin the lesson with, and not her preferred representation for the pupils, indicates a high level of confidence in the textbook. This shows how the textbook exerts a strong influence on her planning decisions. This was something reiterated on several occasions and the quote below is representative of this.

> ***Interviewee****: I think I always say to my staff, particularly the staff that are new in their career, when they come to me and they say 'why am I doing it this way?' I say you're doing it this way for a reason. The textbook is asking them to do it that way for a reason because it's the best way to get the children to understand that concept, so I always say to them 'they're doing it for a reason; what's the reason?' and then they then look down and they realise.*
>
> [Gillian, Interview 3]

Here she is showing the importance placed upon the way in which concepts are presented within the textbook, indicating that it is not just something she considers important for herself, but also for other members of staff. It is interesting to note that at one point in the quote above she refers to the textbook in the third person ('*they're doing it for a reason*') suggesting that the textbook itself maintains some sort of agency according to Gillian. This level of confidence in the textbook also seems to relate somewhat to her belief in using tricky or ambiguous representations.

> ***Interviewee****: It kind of has happened where I've thought 'oh gosh, why are they doing this?' [referring to the textbook] … so I would – I have that conversation with the children sometimes 'right OK in Maths No Problem' it wants us to do this; it wants us to look at this method; what is this method? What's this about? I don't understand it, do you understand it?' and then let's look at and then obviously 'so what's the other strategy?' and then we have conversations as to which is the best.*
>
> [Gillian, Interview 3]

Discussing what happens when the textbook represents things in ways that are very difficult or ambiguous for the pupils, she is suggesting that this could and would still be used to stimulate mathematical dialogue. Again, this shows that she has a great amount of confidence in the design and structure of the textbook and, in this way, is collaborating with the textbook when planning her lessons. It is not just a one-way process - she plans by analysing the textbook, which sometimes challenges her knowledge, and then she makes decisions about how to teach the content. Such confidence in the textbook did not seem to overrule her own professional judgement, however and she did comment that she *'may re-jig around the order'* of chapters at times if she felt the content would fit better for her class in a different order [Gillian, Interview 3]. This suggests that her confidence in the textbook is coupled with a very high level of confidence in her own mathematical knowledge. This is supported by the fact that, within her planning of lessons, she also put in additional challenges for pupils who finished the content before others as the image and field notes below demonstrate.



An image of a pupil's journal with a 'challenge' sticker stuck in by the teacher.

Pupils continue to work quietly writing in their journals. For some, T has stuck in stickers with extra questions for them to consider about the problem they have been doing (see lesson images). Some pupils start to work together to try and solve this.

[Field Notes, Lesson Observation 2]

These examples suggest that, in the case of Gillian, there is a complex interplay between her own professional knowledge and a belief in the textbook as a highly credible resource. This leads to her use of the textbook in a two-way manner. The textbook content will cause her to change her opinions and sometimes do things differently, and sometimes she will change the textbook content to fit with her own knowledge and beliefs, yet despite this, she seems to maintain a high level of commitment to the textbook content.

5.5.4 'Conversations' and representations to understand mathematics

This theme is focussed upon what Gillian refers to as '*conversations*' in her maths lessons, and how she uses this alongside representations in her maths lessons. This was one of the most apparent themes across the whole dataset and it was clear that she placed great value upon the use of talk between pupils, and between the teacher and pupils, in her lessons to help aid the construction of mathematical meaning. The term '*conversation*' has been used as this is Gillian's own terminology, although the type of activity she is referring to would often be described as whole class and small group discussions, or classroom dialogue. It is perhaps testament to the informal nature of some of her teaching practices that she chooses to use the term '*conversation'* instead of any formal educational phrasing. Within the data, she regularly refers to having conversations about mathematical ideas with her pupils and seems to value this highly as a way of helping pupils construct ideas. Much of this is related to her use of representations and she seems to both use conversation as a way of focusing on the features of representations themselves, but also uses representations to stimulate mathematical conversations about different aspects of fractions. In both lessons that were observed, there was a high proportion of talk about representations occurring both between pupils without the teacher, between small groups of pupils and the teacher, and between the teacher with the whole class. It is important to note that this variety of dialogue was also not strictly governed by the teacher and was quite fluid in nature, in this way, this theme is closely connected with the theme 'Balancing pupil autonomy with teacher control'. In both observed lessons it was seen that, during whole-class discussion time, there was still quite a lot of paired talk between pupils, and some continued to draw diagrams whilst the teacher was structuring the dialogue. This observation demonstrates how the lines between what was pupil-to-pupil and what was whole-class conversation were very much blurred, with pupils often breaking off into their own individual conversations during whole class discussions, most of which appeared to be relevant to the main topic. Gillian's main rationale behind this approach was that, by stimulating pupil talk, she is more likely to be able to help them learn about the mathematics she wants them to focus upon. Therefore, although her approach to conversations is quite loose behaviourally, she maintains quite tight control over what is being discussed:

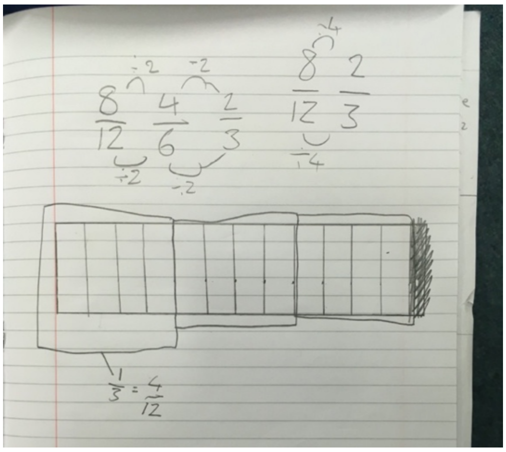
> ***Interviewee****: So, I made sure we're focussing on that. I wanted them to have that conversation about. I know they've read the word 'equal' but I wanted to have that like why has it got to be equal?*
>
> [Gillian, Interview 2]

Gillian uses conversation as a tool to help pupils construct mathematical meaning and, by carefully questioning, she focuses attention on key mathematical aspects within her lessons, prompting pupils to have discussions about these. During a different interview, when discussing her teaching in more general terms, she explains how this approach to conversations is of central importance to what she believes about learning mathematics.

> ***Interviewee**: I do believe in a lot more conversation, start of our… start of my maths lesson in September to set the benchmark in class we have a big conversation and I say to them what's the most important part of this maths lesson and they all come up with do maths, work hard and I say nope that's not the answer, that's not what I want until we get down to conversations, challenging each other – that's what is really important because that then allows you to challenge each other and me to unpick what's going on in your head to tweak or challenge.*
>
> [Gillian, Interview 1]

This suggests that her approach to classroom conversations is something that is actively planned for and that her aim is to build this up as a general classroom expectation. Of key importance here is the interplay between her use of conversations and representations together.

First, it was often the case in the observed lessons that conversations and representations were used together in a symbiotic way – representations supported the conversations, and the conversations supported the use of representations. Frequently, during talk amongst pupils, they would use their own drawn representations to refer to ideas and reason about the mathematics. Below is an image taken from one of the lessons of a jotter where a pupil had drawn some representations during a conversation with another pupil:

 [Field Notes, Lesson Observation 1]

This observation, where pupils are using representations to communicate their mathematical thinking to one another, seems to also be actively modelled and encouraged by the teacher. In the observed lessons it was seen that Gillian would often respond to pupils, who were asking for help, by asking them to draw what they had done and talk though it as they were drawing. Gillian would use this as a way of helping them to communicate mathematical meaning. When discussing one such episode, Gillian reiterated this and explained that she felt, by drawing a diagram, pupils would be able to articulate what they were thinking more clearly:

> ***Interviewee**: I knew he knew it, but he just couldn't explain it enough so I needed him to… by showing me that representation he could talk through it.*
>
> [Gillian, Interview 2]

This approach to encouraging conversation about representations, and using representations to stimulate mathematical conversation, was reiterated as a strong belief in how to teach mathematics by Gillian during all the interviews. In one interview she described it as a *'conscious choice'* that she used dialogue to get pupils *'engaged in the lesson, to be thinking, to see something'* [Gillian, Interview 2]. Gillian sees a strong link between developing pupils mathematical thinking, use of representations and classroom dialogue.

Second, another common feature relevant to this theme was that she would use more than one representation at the same time in her lessons, to help stimulate mathematical conversation.

After looking at the diagrammatic solution, T moves on to looking at the symbolic/abstract solution that some pupils have used (dividing numerator and denominator). At this point, the teaching assistant calls out "Apparently some pupils just saw the 4s…" – this seemed to prompt pupils to make an explicit connection between the symbolic solution to the problem and the diagrammatic one.

[Field Notes, Lesson Observation 1]

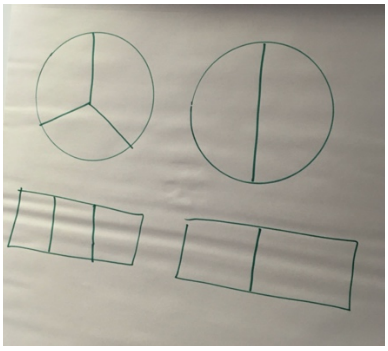
In this example the teacher and teaching assistant are working together to generate dialogue about the connections between a rectangular fraction diagram on the board and a symbolic equation. This sort of interaction where attention was being drawn to relationships between two or more representations was common across both lessons observed. During one of the interviews Gillian explained her rationale behind using different representations and promoting dialogue about these.

***Interviewee**: I needed them to see it and then to build those connections that this can be a variety of different things to deepen their understanding to allow them to explain mathematically…*

[Gillian, Interview 2]

It appears that she believes in using multiple representations in tandem to help pupils deepen their understanding of mathematical objects by discussing the relationships between these, and that conversation about this is her primary way of making this happen. This occurred at several points in both lessons observed but the following episode within one lesson, and the associated interview quote, highlights an additional component of this theme.

The teacher draws attention to the two different representations and discusses with the class the similarities and differences. She asks why you would choose to do the rectangular one and why some might not want to do the circular one.

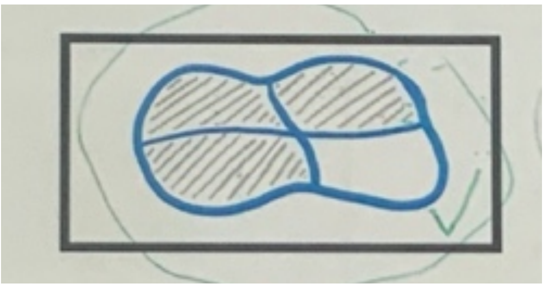


[Field Notes, Lesson Observation 2]

Interviewee*: So here obviously it shows them as pizzas and circles so I knew they'd automatically draw circles but what I needed them to do is not rely on circles because I know that their drawing of circles isn't brilliant, […] I wanted to move them into the bar […] because that's obviously more efficient for them.*

[Gillian, Interview 3]

Within the example above the teacher demonstrates how two different representations are being discussed, yet she believes one to be more useful than the other for the pupils. In this instance the teacher appears to be deliberately using a representation that she believes to be less useful alongside one she believes to be more useful to promote discussion about the mathematical meaning of the problem they were solving. This theme emerged at other points as well, especially during one of the belief and knowledge tasks that formed part of the first interview. One task involved discussing whether she would use a selection of given representations when teaching. Of note, Gillian referred to the representation in the image below commenting that she thought pupils would find this difficult because it is hard to say whether it is three quarters or not. Nevertheless, she also emphasized that she would deliberately use such a representation to provoke conversation that would deepen pupils understanding of fractions.



[Image of problem task 2 with Gillian's markings]

These examples seem to suggest that Gillian holds a belief that tricky or ambiguous representations can, and should, be used to promote mathematical conversation that has the purpose of deepening pupils understanding of mathematical ideas. Here, I am defining representations that the teacher believes to be difficult for pupils due to either their ambiguity or their usefulness as 'tricky or ambiguous' representations. The circular diagram discussed previously is one which the teacher believes to be complex for the pupils and not useful for solving the problem given and the one shown above is believed by the teacher to be ambiguous for pupils. Despite this, the textbook analysis showed that only a limited number of representations were used in the fractions chapter she was teaching (mainly circular and rectangular models or abstract symbols), and these were mainly repeated across the series of lessons. This suggests that perhaps, despite her belief in using a range of tricky or ambiguous representations, perhaps the textbook acted as a buffer to this, limiting the extent to which she did this in practice.

Overall, this theme demonstrates that Gillian believes classroom talk, what she refers to as '*conversation*', is an essential tool for her to use in helping pupils' construct mathematical meaning. Central to this is the interplay between conversations and representations and she believes that they should be used together and was observed to apply this within her classroom practice. This theme relates closely to the following theme about the purpose of representations in maths lessons.

5.5.5 Using Representations for Mathematical Thinking

As the previous theme highlights, using different representations was seen to be a key aspect of Gillian's classroom practice. This theme builds upon the previous, demonstrating how, for Gillian, the use of different representations has the purpose of engaging pupils in thinking about mathematical ideas. Relating to her belief in mathematics as a social, purposeful activity, she advocates an approach to using representations that is driven by purpose. Within this theme there are various elements that relate to this - using representations flexibly, using them to make connections within the mathematics and promoting pupils' own representations. All of these contribute to Gillian's belief and practice of using representations to develop mathematical thinking.

It was very clear from the whole dataset that she placed importance on pupils' being able to use representations in a flexible way. This involves pupils being able to use more than one representation at any one time to communicate thinking and that being able to do this is an important part of deep understanding. In one of the interviews, Gillian watched a video of another teacher teaching about fractions and reflected on how she might have approached the lesson differently:

> ***Interviewee****: I think it would depend on mine… if I knew mine were relying on a bar then I would use a circle but if I knew they were relying on a circle then I would use a bar. Just to challenge it, to bring an extra representation in.*
>
> [Gillian, Interview 1]

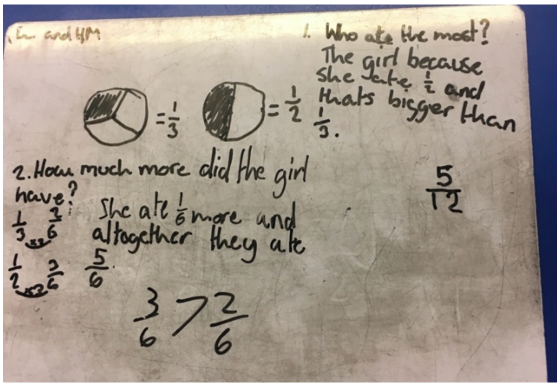
In this quote, Gillian is advocating the use of multiple representations and, specifically, choosing to use ones that move pupils out of their comfort zone. She does not want pupils to become overly reliant on any one representation and is suggesting that she would use different representations deliberately to increase pupils' repertoire. She believes that pupils need to experience a wide range of representations. This use of multiple representations at any one time was something observed in both lessons where, at certain points, three or four different representations were being used together (this included symbolic representations). She devoted specific lesson time to this, where extended discussions took place whilst using the representations, demonstrating the close link between this theme and the theme ‘‘Conversations’ and representations to understand mathematics’. This suggests that this was both a belief and a reality for Gillian.

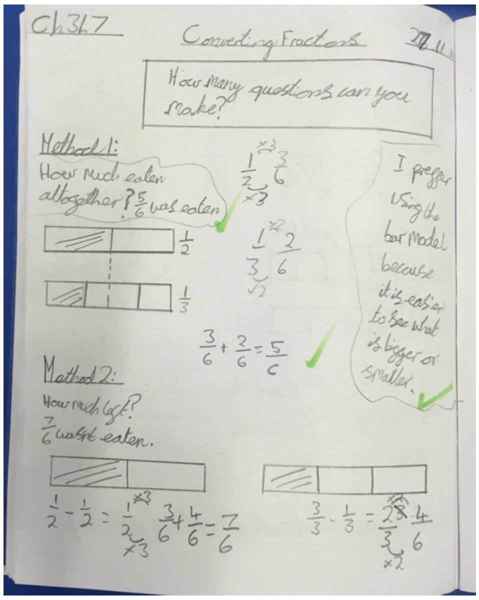
Despite this, although Gillian emphasised that she would use a wide variety of representations during the belief and knowledge tasks (interview 1), she predominantly stuck to using symbolic, language, real life and area model representations in the lessons observed. This is an example of how this theme links with the theme ‘Teacher and textbook collaborating’ because, as the textbook analysis showed, the only representations used by Gillian were those suggested within the textbook (except for strips of paper). In fact, the textbook analysis showed that most representations used within the fractions chapter were limited mainly to real-life scenarios, area models, written English language and mathematical symbols. Therefore, she clearly believes in using multiple representations, however she also opts to use those recommended by the textbook and spend time developing deep understanding of these, rather than introducing a greater number of different representations. It is likely that a more longitudinal study of Gillian’s teaching would reveal different representations being used, as they become introduced within the textbook. The textbook analysis would support this as, within other forthcoming chapters, links are made to fractions and a wider variety of different representations are introduced.

Additionally, her use of multiple representations is not just for the sake of it - she believes in using them purposefully.

> ***Interviewee****: … I want them to build a deeper understanding of it because if it’s just one thing [representation] then they’re not going to actually be able to explore it. It’s like when you see the Maths slapping you in your face with those lower learners, I needed them to see it and then to build those connections that this can be a variety of different things [representations] to deepen their understanding to allow them to explain mathematically.*
>
> [Gillian, Interview 2]

Here, she is explaining the rationale behind her belief in using multiple representations to encourage flexible thinking. She believes that being able to use representations in this way and make connections between them is key to the development of deep understanding. This seems to closely link to her beliefs about pupils as learners of school maths because, again, she is emphasising the importance of using multiple representations over and above simply being able to get answers correct in mathematics. For Gillian, multiple representations are important because understanding the links between them are a fundamental part of what it means to understand mathematics. Alongside this, she also used different representations to scaffold pupils' learning when they are struggling and help them to develop deep understanding. Towards the end of the second observed lesson, pupils were struggling with subtracting compound fractions. Within this part of the lesson, pupils were looking at a textbook question which only had one representation used – an equation using the formal abstract mathematical symbols. Gillian explained that she felt it necessary to *'get back more where we start with a pic representation'* and model the drawing of the problem using a rectangular area diagram (what she refers to as a '*bar'*) because then the pupils would be able to '*see it*' [Gillian, Interview 3]. This suggests that, not only are multiple representations important for deep understanding to her, but that some representations are more useful for making connections within the mathematics than others and that these can be used to scaffold pupil learning. In fact, as mentioned in the theme 'Pupils as Learners of School Maths', the way in which different pupils can use representations flexibly (or not) is one way in which she differentiates between pupils in her lessons. Also, related to this, it is important to note that, within both observed lessons and the subsequent interviews, significant time was given for pupils to represent things for themselves by drawing on notepads, mini whiteboards, and journals.





[Images of Pupil Representations from Field Notes, Lesson Observation 2]

This suggests that not only did Gillian value pupils being able to engage with multiple representations, but that she also valued them being able to represent things for themselves. Most representations created by pupils in the observed lessons somewhat mirrored those shown in the textbook or modelled by the teacher however pupils were not just copying, and it was clear that they maintained ownership over these. Again, this strongly links to her beliefs about pupils as learners of school maths and supports her belief in pupils becoming autonomous learners.

Nevertheless, despite her seeing the flexible use of multiple representations as a way of helping pupils make connections and think mathematically, there was one instance within the dataset that seemed at odds with this:

> *Folding into 12 parts seems an issue – teacher leaves plenty of time for this (10 minutes with discussion). Is this really an effective use of time – doesn't seem to focus on the 'maths'?*
>
> [Field notes, Observation 1]

At the time of writing these field notes, I questioned whether spending ten minutes folding pieces of paper was demonstrating how representations facilitated mathematical thinking. It seemed that this was not a particularly useful way to spend the first section of the lesson (with regards to learning mathematics). However, during the stimulated recall interview afterwards, I questioned Gillian about this decision:

> ***Interviewee****: …some of them will panic because it's a problem. They'll panic; I want to put it into a life context as to when this might happen and break… just to chill out a little minute; just calm down a little minute…*
>
> ***Interviewer:*** *So I find that quite interesting because you're thinking about the way you're using that representation, that piece of paper, not just to purely get them to do the Maths but also to get them to feel something.*
>
> ***Interviewee:*** *Feel comfortable yeah because I think that's a big thing, especially with this class particularly, with quite a few children in the class that if then they don't feel comfortable and we talk about it…*
>
> [Interviewer and Gillian, Interview 2]

It seems to be the case that Gillian also sees the use of representations (in this case folding pieces of paper into equal amounts) as a way of differentiating the emotional environment. She uses the time at the start of the lesson to help the pupils get into a mind state that is conducive to doing mathematics that they may well find difficult, and the role of the representation is key to this as it helps the pupils gradually engage with the lesson content. Although this use of representation is not strictly for the purpose of developing pupils' mathematical thinking, she does suggest that without doing this, pupils would have struggled to engage with the rest of the lesson in a meaningful way. This implies that Gillian considers pupils' mathematical thinking not only in terms of cognition, but also emotion. Her belief is that representations can also be used to make pupils feel at ease when working with mathematical ideas and that this is an important steppingstone into engaging with these concepts in a deeper way. Despite this, she did draw a boundary when it came to using representations for a purpose other than promoting mathematical thinking. During the belief and knowledge tasks, I presented Gillian with a range of representations and asked which ones are appropriate to use when teaching:

> ***Interviewee**: if it takes away from what you are trying to get to, it's just like trying to make something look pretty for the sake of it. You know… if it's got a purpose, use it, but if it's not got a purpose don't use it. If it's going to help you get to something, help you just… get… explore an idea, explore… a concept then use it, but if it's just for the sake of using it, then don't use it.*
>
> [Gillian, Interview 1]

In this quote Gillian is suggesting that all of the representations on the sheet would be appropriate at times when teaching, except for 'images used for decorative purposes'. She is quite firm in her assertion that any representation used when teaching should help pupils engage with mathematical thinking and the observations of her lessons support this. This summarises this theme well – Gillian believes that representations are a powerful way of engaging pupils in mathematical thinking and the lesson observations conducted suggest that this is something that is ingrained into her teaching practice as well.

5.5.6 Summary of Thematic Analysis Themes

This first phase of data analysis involved using a thematic analysis approach, utilising inductive reasoning. In taking this approach, six themes have been identified and explained. The six themes are:

- Mathematics for the people, by the people
- Learning school maths requires resilience and reasoning
- Balancing pupil autonomy with teacher control
- Teacher and textbook collaborating
- 'Conversations' and representations to understand mathematics
- Using Representations for Mathematical Thinking

These provide an important insight into how Gillian thinks about and uses mathematical representations in her teaching of fractions. To further connect the data within this study to the research question and underpinning Theoretical Framework, it is now important to analyse the entire data corpus using the data instruments, as outlined in the Theoretical Framework chapter (section 3.3). Following on from this, the retroductive analysis where Legitimation Code Theory is applied, will be presented.

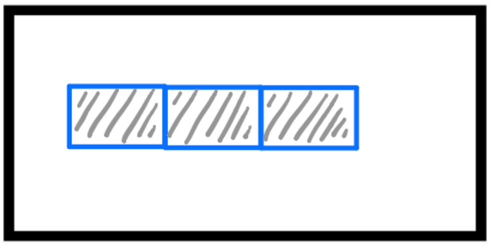
5.6 Section 4 - Thematic Analysis Using the Data Instruments

This phase of analysis involved using the data instruments from the Theoretical Framework chapter (section 3.3). These instruments draw upon the most pertinent literature and help guide the focus of the analysis towards key areas, using deductive reasoning as is discussed in the methodology chapter (section 4.8). Therefore, throughout this section, some reference will be made to key pieces of literature that form part of the data instruments. This will enable some analytical judgements to be made about Gillian's knowledge and beliefs, and use of representations, that are directly connected to previous research. Rather than repeating findings that have already been outlined, this section will refer to the previous section and is thus much shorter in length. Nevertheless, it covers some important key points for this study that need to be outlined. This section is split into two themes that relate to the two data instruments – Gillian's beliefs and knowledge, and her use of representations.

5.6.1 Gillian's Beliefs and Knowledge

Within the first phase of thematic analysis, where inductive reasoning was used, there is already a significant amount of detail about Gillian's mathematical beliefs and knowledge. Specifically, in the themes 'Mathematics for the people, by the people' (section 5.5.1) and 'Learning school maths requires resilience and reasoning' (section 5.5.2), therefore this section will refer to these alongside the data instrument itself. The main finding to report is that Gillian's beliefs and knowledge are availing in nature. This means that the beliefs she holds about mathematics, and the knowledge she has about fractions teaching, are aligned with those that previous research has shown to lead to improved pupil outcomes (section 3.3). First, referring to Kuntze's (2012) model of beliefs and knowledge discussed in the Theoretical Framework chapter (section 3.3), on a global level Gillian seems to predominantly hold beliefs about the nature of mathematics that would fall under Ernest's (1991) fallibilist category. This means that she sees mathematics as a subject that has been created by humans and is there to be used for a variety of purposes, as is further outlined in the theme 'Mathematics for the people, by the people'. However, within the fallibilist category, there may still be a range of beliefs about the subject. When asked about the nature of mathematics at a global level, Gillian suggests that mathematics *'is what you need it to be'* [Gillian, Interview 2]. By saying this, she is suggesting that mathematics is a tool, created by humans, that is there to be used for specific purposes. This seems to be congruous with Polya's (1957: xxxiii) reference to "mathematics in the making", which sees mathematics as being created by humans through social discourse. Second, also at a global level, she strongly promotes mathematics as a social activity and one where being able to communicate thinking is key, another key aspect of a fallibilist belief (Lakatos, 1976). Nevertheless, when we move away from discussing mathematics at a global level, it seems that these beliefs are less strong. During the interviews, when we were discussing mathematics at more specific levels of globality, such as about particular content or instructional situations (Kuntze, 2012), she became more focused on pupils being able to learn about the mathematics she needed them to, as stipulated by the more formal aspects of the curriculum. Her focus was on helping them learn what she saw as the necessary mathematics, but in a way that was still trying to hold true to her global beliefs. Throughout her lessons, she placed a strong emphasis on the social aspects of doing mathematics such as classroom dialogue and communicating thinking. This suggests that Gillian's beliefs about mathematics do fall into the fallibilist category but her global beliefs and knowledge about the nature of mathematics are perhaps at tension with some of the other aspects of her work and the situated beliefs related to these. Her teaching is all taking place within a year group where a national test occurs, and this is used to put schools into league tables nationally. This is something that Gillian specifically mentions and is discussed in the theme 'Learning school maths requires resilience and reasoning'. Therefore, it is perhaps not a surprise that her broad fallibilist beliefs become slightly weaker and more focused on externally determined outcomes for her lessons. It seems likely that Gillian holds a belief system (Thompson, 1992; Leatham, 2006) where fallibilist beliefs are important, but where a belief in ensuring her pupils do well on their end-of-year-tests and learn specialised mathematical knowledge is also important, and these two beliefs combine and influence the way she teaches her lessons. It is also important to note that Gillian holds a strong belief that pupils needed to learn about the interconnected nature of mathematics and that this was tangled up with her belief in mathematics as a social subject and her use of representations. For Gillian, mathematics should be a purposeful subject and, to enable this in her practice, she felt that pupils needed to see the connections between ideas. This was evident at all levels of globality – she expressed it as a general belief, but it was also evident in her teaching where she strongly encouraged pupils to make connections between representations and thus understand the concepts she was teaching in a deep and meaningful way. In fact, in the theme 'Learning school maths requires resilience and reasoning', it is evident that she saw the ability to make connections between and use multiple representations for problems as a key part of what it means to be good at school maths.

As well as analysing Gillian's global beliefs and knowledge of mathematics, the theoretical data instrument also guided attention to her content domain-specific beliefs and knowledge (Kuntze, 2012), specifically about fractions, due to their central place in this study. First, it was clear that Gillian had what I refer to as a 'confident knowledge' of fractions, meaning that she was not hesitant to talk about them and could analyse different representations from a teaching perspective. This puts her at slight odds with previous research that has highlighted teachers' difficulty with fractions (Askew et al., 1997; Ma, 1999), and raises the question as to whether she is simply an 'odd one out', or whether the research previously conducted into this area perhaps shows an out-of-date picture of teachers within England. Specifically, Gillian was comfortable with the five different fraction sub-constructs (Kieren, 1976) when doing the teacher problem tasks interview. For example, she could comfortably identify them all as different representations of the fraction '$\frac{3}{4}$'. Despite this, she did have some difficulty with the division of a fraction by another fraction and got slightly muddled during the interview. Later, during a follow up interview, she was keen to point out and demonstrate that she could do division with fractions, but that this was not part of the formal school curriculum that she taught, hence she was a little out of practice. In addition to this, her understanding of the different possible representations of fractions was very comprehensive and she was confident working with and discussing almost all different representations and how they were connected and might be used for teaching. The textbook analysis showed that a wide range of representations were used within the book, specifically when making links in other chapters (E.g., in measures, percentage and ratio) and this is perhaps one reason why she was so confident in her knowledge. Given that the resource she was using every day to plan and teach lessons presented a wide range of representations, it seems likely that this went some way towards influencing her knowledge and beliefs about them. The only exception to her knowledge in this area was the example of iterating fractions. When presented with the image below, she struggled to see how this could be '$\frac{3}{4}$' (this requires seeing it as three quarters with one quarter removed).



[Image from the teacher problem tasks]

Interestingly the textbook analysis showed that there were no examples of iterating, and this is perhaps one explanation for her difficulty. Another important dimension to her knowledge and beliefs about fractions was the way in which she used fractions in her teaching. Although during the teacher problem tasks, she confidently talked about a very broad variety of fraction representations and how they could be used, in her actual lessons, she only used a small number of representations. This suggests that she also had a good understanding of how pupils learn fractions and the need to not overload with many different fraction structures all at once. Of note, the textbook analysis also showed that the way fraction constructs were introduced was slow and most of the fractions chapter relied upon just a few core representations that are in line with research, showing which ones are most effective when learning about fractions at an early stage (Tunç-Pekkan, 2015). In summary, Gillian had a confident knowledge of fractions and their associated representations, and this seems to be supported by the textbook scheme that she uses. This section has already alluded somewhat to the way in which Gillian uses representations through discussion of her knowledge about them, and the following section provides further detail in this area.

5.6.2 Gillian's Use of Representations

The way in which Gillian uses representations is covered quite comprehensively within the first phase of thematic analysis, specifically in the themes ''Conversations' and representations to understand mathematics' (section 5.5.5) and 'Using Representations for Mathematical Thinking' (section 5.5.6). Therefore, here these two themes will be referred to in relation to the theoretical data instrument and presented in a concise manner to avoid duplication. First, although it may be obvious from the previous section, Gillian does use *multiple* representations in her teaching, with a focus on using them to develop mathematical meaning. Notably, when discussing representation in general terms during the teacher problem tasks, she espoused the importance of using many different types of representation, whereas in her observed lessons, she stuck to just a small number. However, analysis of the textbook showed that, if Gillian was to teach all that was in the textbook throughout the school year, then a wider variety of representations would have been used over a longer period (due to different representations being introduced in chapters related to fractions such as decimals, measure and percentage). Therefore, it seems reasonable to suggest that Gillian's belief in using a broad range of representations would become a reality but is perhaps being guided somewhat by the progression within the textbook scheme she is using. It is also important to note here that, when using different representations, Gillian is clear that any representation used should have a precise purpose, as is outlined in the theme 'Using Representations for Mathematical Thinking'. For her, the purpose of using any representation(s) in a lesson is to help pupils develop deep mathematical knowledge and those designed purely for cosmetic purposes should be avoided. This aligns with the research findings and suggests that Gillian's use of representations may lead to a positive impact on pupil learning in the long run (Carbonneau, Marley and Selig, 2013). Nevertheless, there were times within the data that Gillian did seem to use representations for something other than mathematical learning, and this was to help pupils get into a positive mindset about fractions. This suggests that perhaps Gillian treated representation as a broad concept, which includes attitude towards mathematical concepts (specifically fractions), something that was less prominent in the literature but still an important element worth noting (Goldin, 1998, 2002b). Second, as outlined within the theme ''Conversations' and representations to understand mathematics', Gillian also treats representations as objects for discussion and allows pupils a large proportion of lesson time to verbally reason about the representations they are using. Some instances within the observed lessons, where pupils start to use diagrams to reason about their ideas without teacher prompting, suggest that this use of dialogue alongside representations has become part of the culture within Gillian's classroom. This is something that Gillian talks about actively trying to cultivate from the beginning of each school year with the classes she teaches. This also aligns with the research literature and strengthens the argument that the way Gillian uses representations is likely to lead to improvements in pupil learning over time (Carbonneau, Marley and Selig, 2013; Rau and Matthews, 2017). Finally, another aspect of representation use that was less evident in the literature but still important was the provision of opportunities for pupils to develop their own drawn representations (Meira, 1995). Although this was not something that Gillian talked about as important to her in the interviews, in the observed lessons she provided pupils with numerous opportunities to draw representations both informally (using mini-whiteboards and jotters) and formally (as recorded in their maths journals). Nevertheless, it is hard to say the extent to which this was pupils creating their own representations, or just imitating ones used by the teacher in previous lessons. Most pupils used either circular or rectangular area models in their drawings and these are also the ones that were being used by Gillian and that the analysis showed were most prominent in the textbook scheme across all year groups.

In summary, it seems that, according to the data instruments, Gillian's mathematical knowledge and beliefs, and approach to using representations are aligned with what the literature would suggest is effective practice. In the next section, the retroductive analysis will be presented, showing how Legitimation Code Theory was applied to the data to help ensure a strong connection between the empirical data and the underpinning Theoretical Framework (3.3).

5.7 Section 5 – Retroductive Analysis Applying LCT Dimensions

For this phase, the LCT dimensions of Specialization and Semantics have been used to analyse the data using a retroductive approach. Using a retroductive approach in this section is about providing explanations for the explanations presented in the previous phases of analysis and getting under the surface of the data (Scott, 2010; Maton, 2016). Analysis in relation to the Specialization dimension will be presented first, followed by Semantics.

5.7.1 Analysis Applying the Specialization Dimension

The Specialization dimension is important to this study because it focuses attention on the extent to which knowledge, and ways of knowing, are strongly or weakly emphasised in a particular social situation. In the case of Gillian, it should enable a deeper understanding of what she believes to be legitimate knowledge and ways of knowing in in relation to school maths. Therefore, the data used for this part of the analysis is primarily from the interviews, however some reference is made to what was observed in her lessons as well. Importantly, the data analysis here was done through identifying and then coding ‘segments’ of pertinent data from the whole data corpus. I define ‘data segment’ for this phase of analysis as being either an extended quotation from an interview or, at times, an extended quotation placed alongside relevant field notes. First in this section, four data segments will be presented with ensuing written analysis. These will be chosen to demonstrate the scope of variation within the data corpus in relation to the Specialization dimension. Following this, my analysis of all data segments will be presented visually on a cartesian plane with ensuing written explanation. Thus, the four exemplar data segments provide transparency as to how I have interpreted the Specialization codes in relation this study. In this way, the four examples act as a “translation device” so that the reader can gain a better understanding of how the empirical data has been translated into the Specialization codes (Maton and Chen, 2016: 43). In figure 19 below, each of the four data segments have been placed onto a cartesian plane to provide a visual representation of how each has been coded.

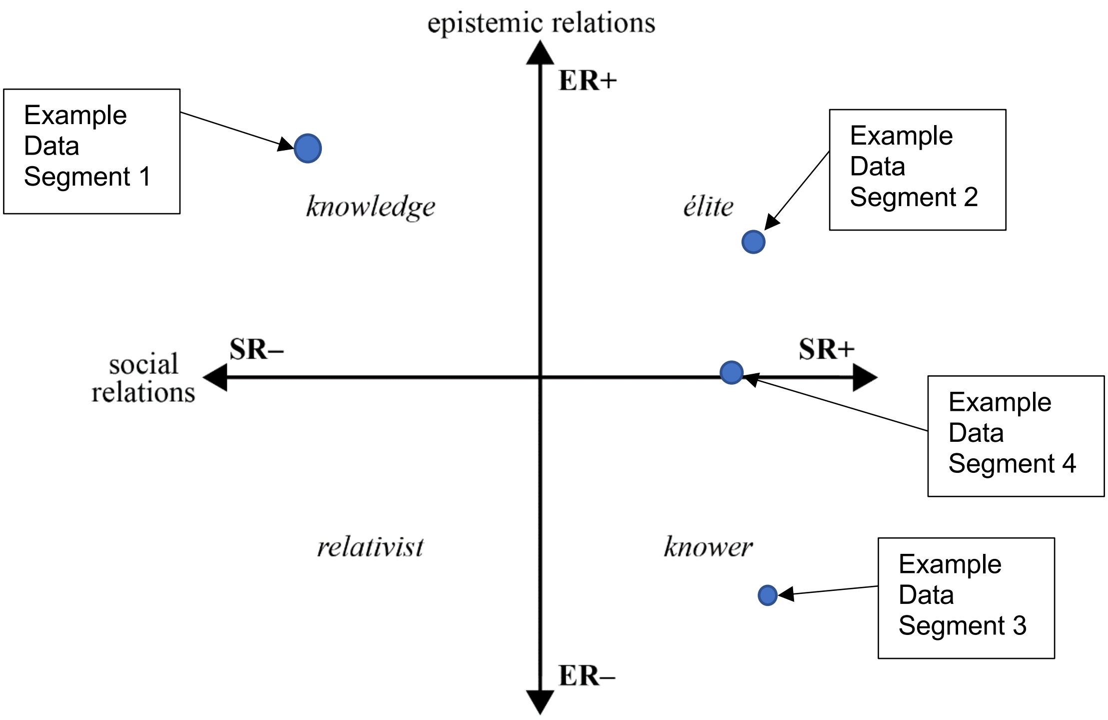


Figure 18 - Exemplar data segments plotted onto the Specialization dimension

Example Data Segment 1 – An example of a 'knowledge' code within the data

Although it was rare within the whole data corpus, there were times where Gillian displayed a 'knowledge' code. In the following data segment, she is responding to a point in the second observed lesson where pupils were struggling, and she had started to tell the pupils what to do and demonstrated this using a rectangular diagram.

> ***Gillian:*** *Yeah well the first – well we've hit that one anyway because I knew that was going to be coming, that's something that had to solve; one other child did come up with that but this young man here said 'there's a sixth left' so he actually didn't get it right so I wanted – and he didn't like the fact that I said it's wrong. He went and he actually heckled me and went 'yeah it is right' which, at that point, it was 'no it actually is wrong and I'm going to tell you it's wrong but I want you to find out why it's wrong' and then obviously, you know, it was because he'd forgotten that there were two pizzas that he'd taken from so we looked at that.*
>
> [Gillian, Interview 3]

> *At this point the amount of pupil talk is very limited and most pupils seem to be quietly listening to and looking at the teacher. This part of the lesson is quite different to any other part due to the limited pupil talk and the teacher demonstration – perhaps this is related to the fact that the pupils where struggling.*
>
> [Field notes, Lesson Observation 2]

I have coded this segment as demonstrating a 'knowledge' code because it is an example where Gillian is promoting a specialised piece of mathematical knowledge over and above certain social behaviours that she had previously suggested were important to her teaching (ER+, SR-). Although in a previous interview Gillian had strongly promoted pupil discussion and getting them to '*heckle*' her [Gillian, Interview 2], here she is downplaying this and opting for explicit teacher explanation to the extent that she cut a pupil off whilst they were trying to heckle her, so that she could explain the mathematics. The lesson observation notes support this showing that, at this point in the lesson, there was considerably less pupil talk and more teacher demonstration of the mathematics and this was at odds with most of what was seen. Importantly, as is mentioned in the field notes, this occurred at a time when pupils seemed to be struggling with the mathematics. Although there were not many segments of data coded as representing a ‘knowledge’ code within the data corpus, this was the only point in either observed lesson where most of the class were struggling with the mathematical content. It is possible that this type of occurrence (where most pupils are struggling) is a trigger for Gillian to downplay her beliefs about social relations (SR-) in lessons and raise the importance of pupils learning the specialised knowledge (ER+) that is her chosen focus for the lesson.

Example Data Segment 2 – An example of an ‘elite’ code within the data

Within the data corpus, the segments analysed showed quite a few instances of an ‘elite’ code. In the following data segment Gillian is using a specific pupil’s own representation as a model for the class specifically to ‘*boost his confidence*’ [Gillian, Interview 2]. Nevertheless, she is also wanting to use it as a teaching point for some specialised knowledge integral to her lesson, so she decides to re-draw the diagram herself and talk through it, with the whole class listening.

> ***Gillian:*** *And that’s kind of why I chose their board as well because I wanted him to be like ‘look your representation is a fantastic one and I want you to be proud of that and I want us to use that as a teaching point’ but to be fair most of the others had got that anyway so I could have picked on a few other children but I chose him specifically because I wanted him to see it get that. I was trying to massively boost his confidence at that moment.*
>
> ***Interviewer:*** *So that’s where you show his – an image of his whiteboard to all the class whilst having a discussion and it’s interesting because I noticed at this point – I’ll play it as we’re talking – you then here have moved from looking at his whiteboard and talking about it to then you almost draw your own version of it so tell me a bit about that because you’ve gone from the child’s own version of it to his image of it to then your teacher I think version, drawn diagram of his diagram, does that make sense?*
>
> ***Gillian:*** *Well two reasons. The easiest, simplest one is because it was on the side. That’s the easiest one but that’s not the reason. It was more that I wanted to go*

through the process of it because his was the end product so, you know, with the colours drawn on it whereas I wanted to go through that and slow it down to 'oh he saw those 2s' so that we could then draw out where actually was that and how because someone's end product can be really confusing if you've not understood the process so I wanted everyone to see the process, particularly her, you know, one of our really lower learners, I wanted her to see that process that was going through, of how they use that representation to get to their answer.

[Interviewer and Gillian, Interview 2]

Gillian brings everything together – "does anyone want to explain to me how they found out?"
As well as looking at pupil's own representations, she also models a neater version of how the bar model could have been drawn in response to what pupils are telling her.

[Field Notes, Lesson Observation 1]

I have coded this segment as demonstrating an 'elite' code because it is a good example of how Gillian balances her belief in cultivating what she believes to be important personal traits, alongside her precise teaching of specialised mathematical knowledge. This type of instance was quite common within the data corpus and her teaching seemed often to rely on a careful balance of developing pupils' personal attributes alongside helping them learn specialised mathematical knowledge that she has chosen to focus upon, influenced by the textbook she is using. In this data segment, Gillian is strongly emphasising social relations (SR+) because the whole reason for her selecting a particular example from a pupil is to boost confidence and build up a positive feeling towards mathematics. She could have chosen other, better, examples from different pupils, but she explicitly says that she decided to use her chosen example for social reasons. Therefore, this is an example of how Gillian's actions are based strongly on developing social relations. Nevertheless, she does not stop there. She then decides to draw her own version of the pupil's representation and talk the whole class through the process. This aspect of the data segment shows how Gillian is also strongly basing her actions on epistemic relations (ER+). Whilst boosting the confidence and attitude of one pupil, she is also acutely aware of the need for the whole class to understand the mathematics that is the focus of her lesson. Therefore, this segment represents an 'elite' code because she is strongly valuing both social and epistemic relations.

Data Segment 3 – An example of a 'knower' code within the data

As part of the final interview I did with Gillian, I asked her about what she thought it took for a pupil to succeed in school maths:

> ***Interviewer:*** *The question was… What do you think it takes for someone to be… for a pupil to be good at maths in school… what… what do you*
>
> ***Gillian:*** *//Ahh! Resilience. It takes resilience.*
>
> ***Interviewer:*** *Go on, tell me more.*
>
> ***Gillian:*** *Resilience and… and… an open mindedness to try and its resilience to call on the things that they need as opposed to mathematical ability.*
>
> ***Interviewer:*** *Ok.*
>
> ***Gillian:*** *//They need to have that kind of mindset that, you know, 'I can have a go at this, I can do it". And then they have that logical thinking to work through it. Um… That's more important than anything because a gifted kid that can just do maths really quickly and doesn't have resilience struggles… So, for me it's resilience that's the most important thing. Completely… Erm… I suppose then they need… do they need? Or… Do they need to be given?*

[Interviewer and Gillian, Interview 4]

Within this data segment, Gillian is clearly emphasising that, for her, it is social relations such as resilience and having a positive mindset that are the basis for success in school maths. She interrupts my question to talk about the importance of resilience which suggests that this is something she feels quite strongly about. Additionally, she also voluntarily contrasts this with '*mathematical ability*' [Gillian, Interview 4], suggesting that here she is valuing social relations over and above epistemic relations (SR+, ER-). For her, if a pupil does not have resilience, then they will struggle to succeed in school maths, despite their ability to learn the specialised knowledge that the curriculum consists of. I have coded this as a 'knower' code because of this – Gillian is downplaying the importance of specialised knowledge in favour of certain personal traits. Importantly, she is not emphasising personal traits that are fixed or inherent within pupils, she seems to be promoting these as something that all pupils can develop and the previous phases of analysis show that she clearly adapts her teaching to provide opportunities for this. There are two important points to raise about this data segment. First, it is a more extreme example of a ‘knower’ code within the data corpus and, most other segments, had a stronger epistemic relation than this one. Second, this data segment was quite detached from Gillian’s daily practice. It was not part of a stimulated recall interview, and we were not talking about the specifics of her classroom practice. This perhaps goes some way to explaining why this excerpt downplays the importance of specialised knowledge as the basis of success more so than others that were more grounded in the specifics of her classroom practice, such as from the stimulated recall interviews.

Example Data Segment 4 – An example of a data segment close to the boundary between two codes

This final example of a data segment has been chosen because many of the segments analysed were not as clear cut as the previous three, in terms of coding. Many segments fell either into the ‘knower’ or ‘elite’ codes but were close to the line. The example data segment below is one such example where the lines between these two codes might be blurred slightly. This segment refers to the start of the first observed lesson where Gillian asked pupils to fold strips of paper into twelfths to represent an image from the textbook. She allowed them ten minutes for this and, as my field notes indicate, it seemed like the pupils were not particularly focussed on mathematical thinking during this time. For transparency, it is important to point out that this data segment does contain a much longer part of the transcription but here, I have cut two of the most pertinent sections out to illustrate the way in which I have coded the whole segment.

> *Folding into 12 parts seems an issue – T leaves plenty of time for this. Is this really an effective use of time – doesn’t seem to focus on the ‘maths’?*
>
> [Field Notes, Lesson Observation 1]

Interviewer: So I find that quite interesting because you're thinking about the way you're using that representation, that piece of paper, not just to purely get them to do the Maths but also to get them to feel something.

Gillian: Feel comfortable yeah because I think that's a big thing, especially with this class particularly…

[Interviewer and Gillian, Interview 2]

Gillian: Yeah I think it would have been nice if I'd had more – to get that to the success, let's do it together but because I was very aware that we'd already spent what 10 minutes folding pieces of paper I thought right actually we need to move on here a little bit, we need to actually get to the Maths

[Gillian, Interview 2]

When analysing this segment, it was clear to me that Gillian's use of the pieces of paper was an example of her emphasising strong social relations (SR+). Although asking the pupil to fold the paper into twelfths was relevant to the specialised content of the lesson, she acknowledges in the interview that her choice of allowing ten minutes of lesson time to do this was to help pupils get comfortable with the lesson content and into a more positive mindset (SR+), rather than to help them succeed in learning the mathematics. At first, I felt that this demonstrated a strong 'knower' code because it seemed like she was downplaying learning the mathematics as the basis of success. However, in the second interview excerpt above, Gillian describes how she felt about the length of time showing that she felt it important to '*get to the maths*' [Gillian, Interview 2]. This suggests that, within this segment, she is not entirely downplaying the importance of the pupils learning specialised knowledge and getting onto this within the lesson was important to her to ensure success. Therefore, I felt that it represented neither a strong nor a weak epistemic relation (ER+/-). This instance demonstrates how, at times, balancing the emphasis on learning specialised mathematical knowledge in her lessons with her strong focus on social relations was tricky for Gillian and many of the data segments fell somewhere between an 'elite' and a 'knower' code.

Summary of Analysis Applying the Specialization Dimension

The four example data segments above are designed to provide a guide to help the reader understand how each data segment has been translated into a code within the Specialization Dimension. The figure below (figure 20) illustrates how all the data segments from the entire data corpus were coded. Although the four examples provide explanation of different aspects of Gillian's beliefs and practices, it is useful to draw some overall conclusions about these in relation to the Specialization dimension before moving on to analysis using the Semantic dimension of LCT.

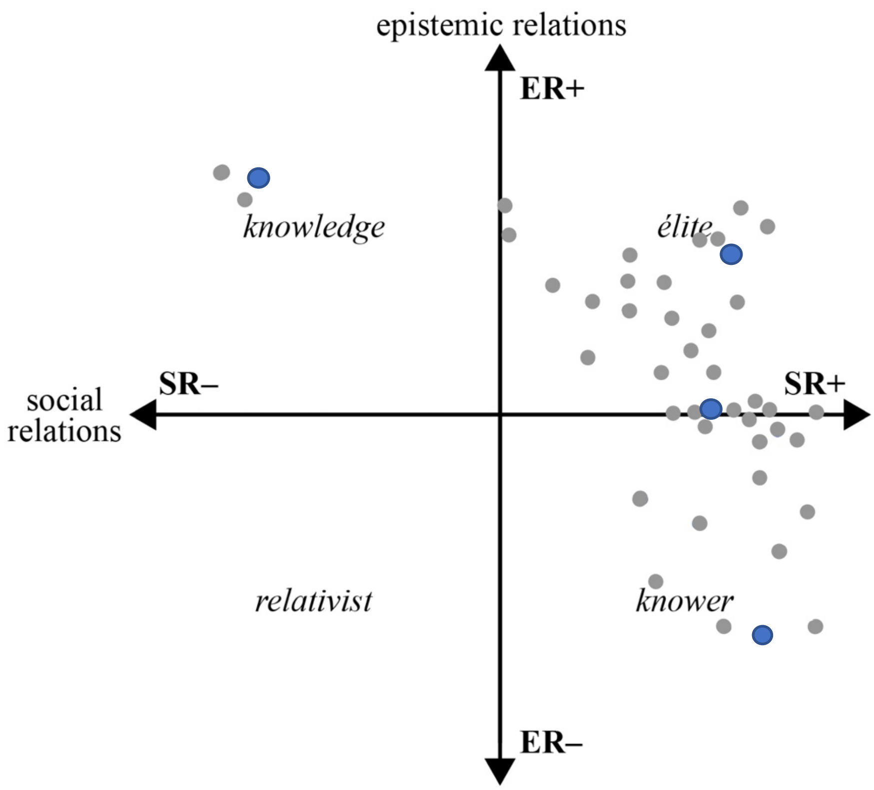


Figure 19 - A representation of all data segments plotted onto the Specialization Plane (exemplar segments in blue)

As can be seen within figure 20, most data segments were coded as being representative of either a 'knower' code, an 'elite' code, or somewhere in-between the two, as example data segment four demonstrated. This suggests that Gillian's beliefs and practices involve a delicate balancing of the learning of specialised mathematical knowledge, with the development of particular social traits and behaviours, as the basis of success in her maths lessons. Importantly, which one she emphasises as the basis of success more, depends upon her perception of the needs of the pupils. For example, within the data segments, she would downplay the importance of epistemic relations (ER-) and really promote certain social relations (SR+), leading to a 'knower' code, when she deemed pupils to be lacking in confidence or, at times, over-confident. It is as if, during these occurrences, the focus of her lesson has shifted and her main aim for the pupils moves away from learning a piece of specialised mathematical knowledge and moves towards the development of a specific personal trait or behaviour. For example, the previous phases of analysis highlighted that she strongly valued pupils developing resilience, a positive mindset towards maths, the ability to learn through dialogue, and autonomy in their learning. All of these are personal traits or social behaviours that, when considered as fundamental to success, lead to either a 'knower' or 'elite' code within the specialization dimension. It is interesting to also note that the thematic analysis in section 2 in this chapter (within the theme 'Balancing pupil autonomy with teacher control') highlighted that these social traits and behaviours were often not things that she kept tight control of. This suggests that Gillian's approach to helping pupils develop these social relations was much looser in terms of teacher control, when compared to her control over pupils' learning of mathematical knowledge. Nevertheless, it is important to point out here that her emphasis of these things was for the purpose of helping pupils become better learners and thus more competent with the specialised mathematics that she was teaching them. As can be seen in figure 19, many of the data segments showed that Gillian also valued the learning of specialised mathematical knowledge as the basis for success (ER+). Gillian was quite specific about the type of mathematical knowledge she felt was important for success in her lessons. As can be seen in more detail within the thematic analysis (section 5.5), she strongly promoted pupils being able to use representations to explain concepts, make connections between different representations (including abstract symbols) and solving problems in more than one way. It is especially important to note that Gillian was quite clear that it was her job, as the class teacher, to decide the mathematical focus of her lessons and she was strongly influenced by the textbook in this process. She kept quite close control of this aspect of her practice and felt that it was her own confident subject knowledge, combined with the textbook, that enabled her to do this effectively. Interestingly, in the few instances where the data segments fell into a ‘knowledge’ code, it was either when Gillian was talking about how she planned her lessons with the textbook, or when pupils in the lesson were struggling with the mathematics (as in example data segment 1). This is important as it again shows how, the basis of success within Gillian’s practice shifts depending on her perception of pupil needs, but also on the influence of the textbook in this instance.

Overall, this analysis applying the specialization dimension shows how the basis of success in Gillian’s maths lessons involves both the development of social traits and behaviours, but also developing a deep understanding of specialised mathematical knowledge, as dictated by Gillian, in collaboration with the textbook she uses.

5.7.2 Analysis Applying the Semantic Dimension

The Semantic dimension of LCT focuses upon the complexity (Semantic density) and context dependence (Semantic gravity) of meaning, and how it is communicated between people (Maton, 2016). It is important to this study because the way in which teachers use different representations to communicate meaning in the classroom is a central part of my research question. Therefore, by conducting retroductive analysis on Gillian’s use of representations, applying the Semantic dimension of LCT, the aim is to shed further light on how meaning is communicated in her school maths lessons. Because this aspect of the analysis is solely focussed upon the communication of meaning in her teaching, the primary sources of data used for analysis are the lesson observations and textbook analysis, which provide real examples of how Gillian does this. The analysis of the lesson observations has been done by using the segments identified in the lesson descriptions and then coding these using the Semantic dimension. The textbook analysis has been used to support this by analysing the content of each lesson. First, in this section, three examples of the lesson segments and how they have been coded will be presented. This is so to demonstrate the range of variation within the data in relation to the Semantic dimension. As with the previous section, these three examples are designed to provide transparency as to how I have interpreted the data and act as a translation device for the reader (Maton and Chen, 2016). In figure 21 below, these three segments have been placed on a cartesian plane, showing the four codes within the Semantic dimension. Following on from this analysis, one of the lessons (lesson 1) has been represented as a 'Semantic wave' (Macnaught, Maton, Martin and Matruglio, 2013) to demonstrate the temporal nature of the way in which Gillian uses representations in her lessons. Finally, a summary analysis of all the lesson segments will be presented on a cartesian plane in order to show the full scope of Gillian's teaching in relation to the Semantic dimension.

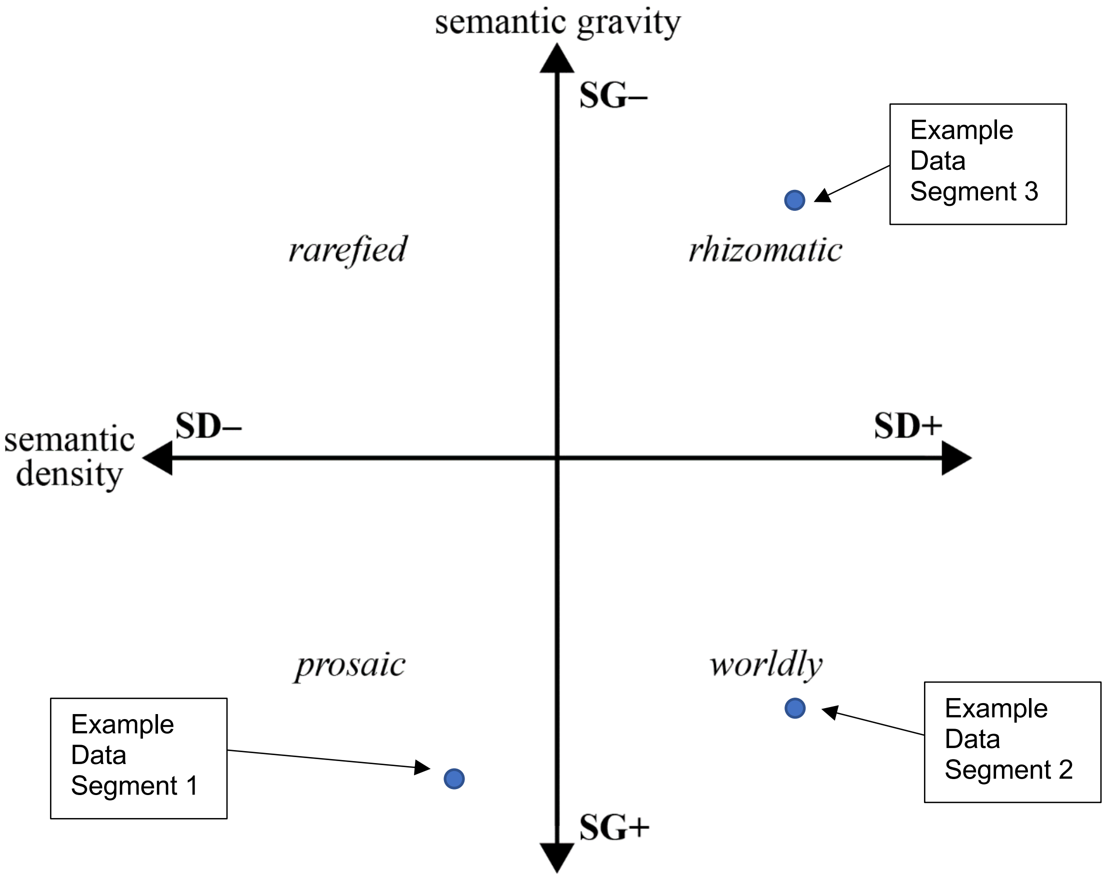


Figure 20 - Exemplar data segments plotted onto the Semantic dimension

Data Segment 1 – An example of a 'prosaic' code

The following lesson segment is taken from the beginning of the first lesson I saw Gillian teaching. The general description of this lesson segment, along with the image used on the interactive whiteboard is shown below.

1. Beginning the lesson	Gillian begins the lesson by showing an image from the textbook on the screen (of a jam roll split into 12 equal parts). She provides pupils with white strips of paper and asks them to imagine that it represents the jam roll and to fold their paper in the same way the jam roll has been split up into 12 equal parts. Pupils take quite a lot of time doing this and discuss what they are doing amongst one another.	10 mins

[Extract from Lesson 1 Description]

This is the image that Gillian had presented on the screen during the described lesson segment above.

[Lesson Observation 1 Field Notes]

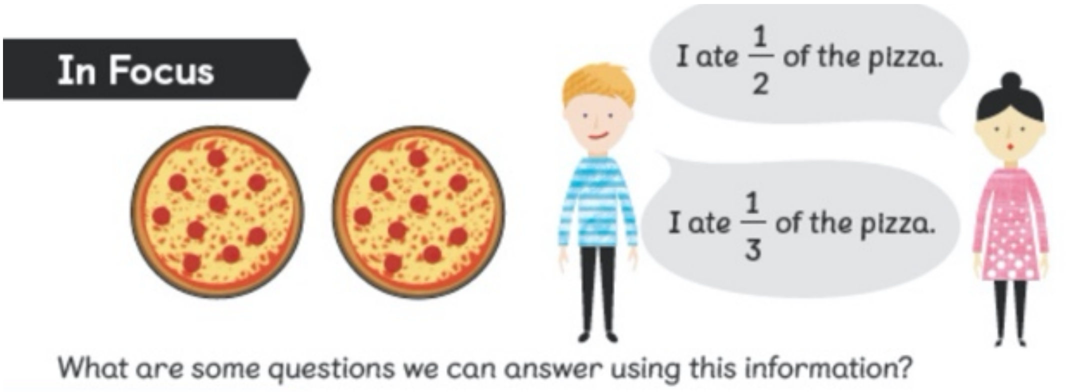
In this lesson segment the meaning being negotiated between teacher and pupils is highly context dependent. The focus in the lesson is on a problem, that is based upon a possible real-life scenario, which is about a jam roll. The teacher shows the pupils the image but then asks them to use strips of paper and to imagine that these are the jam roll. In this sense, the meaning being negotiated is rooted in a fictional real-life scenario, but the teacher is creating a connection to a physical representation (strips of paper) that the pupils can manipulate. Alongside these two representations, the teacher uses the phrase '*twelve equal parts*' [Lesson Observation 1, Field notes]. This makes the communication of meaning quite low in semantic density (SD-). Although three different representations are being used (a jam roll image, paper and verbal language to describe the equal parts), there are still very few connections made to any other aspect of mathematics other than the basic idea of splitting something into equal parts. Alongside this, the representations being used are bound to the real-life context and all the classroom discussion about the image and strips of paper involved constant reference to the jam roll scenario. Therefore, this lesson segment also has a high level of semantic gravity (SG+). Interestingly, this was the only lesson segment that I coded as ‘prosaic’, and this can perhaps be explained by the inherently interconnected nature of mathematics as a subject discipline. For example, if Gillian had introduced the idea that this jam roll represented a fraction and labelled it with something like ‘$\frac{12}{12}$’ on the screen, then the semantic density of the segment would have become positive, making it a ‘worldly’ code instead, as the number of possible connections that the pupils would be making would significantly increase. Nevertheless, Gillian seems to deliberately avoid doing anything like this at this initial stage in the lesson, opting to keep the representations used as simple as possible and ensuring that the focus is rooted in a simple real-life scenario. It is as if she wants the pupils not to over-think things at this point.

Data Segment 2 – An example of a ‘worldly’ code

The following lesson segment is from the second part of lesson two and it follows on from Gillian showing an image to the pupils (shown below) and discussing what possible mathematical questions could be asked of it.

2. Pupils’ own problem creation	After this, Gillian then asks the pupils to think about the different questions that could be asked with this information, and they start collaboratively working on this in pairs. Pupils use their jotters to draw a variety of diagrams and use some symbolic representations. Most pupils come up with problems involving the addition of fractions. There is a lot of loud pupil discussion.	7 mins

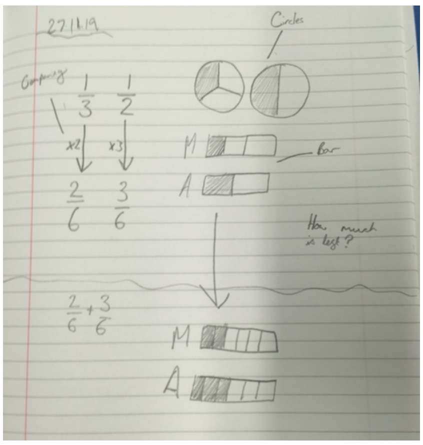
[Extract from Lesson 2 Description]



This is the image that Gillian was showing on screen during this lesson segment.

[Lesson Observation 2 Field Notes]

Within this lesson segment, the teacher is asking the pupils to think about the fictional scenario given in the textbook and consider creating their own questions about it. Because the whole segment is focussed on the scenario of two people eating some pizza, it is still very much tied to a real-life context, therefore there are high levels of semantic gravity (SG+). Supporting this, during the observation, the discussion between pupils was generally focussed on coming up with questions about these two characters and the pizza. Nevertheless, despite this type of discussion, pupils did use abstract symbols and different diagrams (not just circles to be the pizzas) in their jottings:



An example of a pupil's jottings during this lesson segment. These show no reference to the real-life context, despite the nature of the discussion being had.

[Image taken from Lesson Observation 2 Field Notes]

This suggests that, although pupils were referring to the given context, they were able to use context-independent representations, which they had created themselves, to describe and discuss the context. This suggests that, although there are still high levels of semantic gravity, this segment is perhaps a little less context-dependent than the previous one. Additionally, the use of the abstract fraction symbols here, alongside different representations, mean the complexity of meaning being communicated is much higher and therefore the semantic density is higher (SD+). The combination of high semantic gravity and high semantic density (SG+, SD+) mean that this lesson segment has been coded as ‘worldly’. Of interest, as the seven minutes of this lesson segment progresses, there is less mention of the real-life scenario and discussion becomes more about the abstract fractions. This suggests that, through time, the amount of Semantic gravity is reducing within this lesson segment. This shows the importance of tracing the Semantic gravity in a temporal nature, and the analysis using the Semantic wave in a subsequent section will demonstrate this.

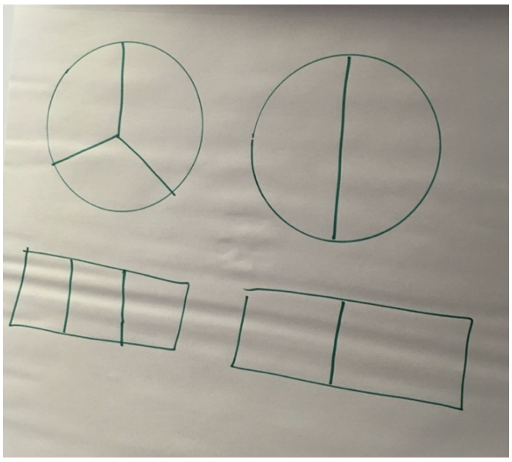
Data Segment 3 – An example of a ‘rhizomatic’ code

The following segment is taken from a later point in the second observed lesson. The pupils have been struggling with the lesson content and Gillian decides to reflect on what they have been doing with them and talk it through.

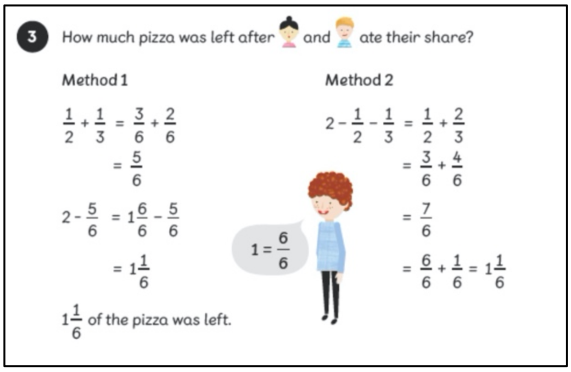
5. Reflecting and summarising	Moving on, Gillian then states that she thinks that the pupils need some time to reflect on what they have been doing. She slowly talks through the ways in which they have discussed the subtraction of two subsequent fractions from a whole number. During this, she refers to the circular, rectangular and symbolic representations. There is no mention of the original problem context (pizzas) at this point.	4 mins

[Extract from Lesson 2 Description]

At this point in the lesson, Gillian has moved almost completely away from the original real-life scenario of the textbook problem. Instead, the focus of discussion is on the abstract fractions and the different methods and representations that can be used to help pupils complete the subtraction equation “$2 - \frac{1}{2} - \frac{1}{3} = ?$” This means that the semantic gravity has significantly decreased (SG-) within the lesson by this point and the communication of meaning is much less about a real-life scenario and more about the manipulation of abstract mathematical ideas. Additionally, within this segment, Gillian is using a range of different representations as can be seen in the lesson images below.



Gillian's own diagrams, drawn on a flipchart, that form the basis of whole class discussion during this segment.



This is the image from the textbook that Gillian was showing on the screen during this segment that was also referred to during the discussion.

[Images taken from Lesson Observation 2 Field Notes]

Alongside these representations, during the class discussions, both pupils and Gillian use spoken language to refer to the fractions. The way in which Gillian used the different representations, making connections between them, means that the Semantic density in this lesson segment was high (SD+). The ability to communicate meaning using the abstract symbols for fractions alone is an inherently complex aspect of mathematics. When combined with the expectation that a connection can be made with both the rectangular and the circular area models, along with the process of subtraction, this segment presents a highly complex and therefore Semantically dense part of the lesson. This leads to the coding of this segment as ‘rhizomatic’.

Tracing the semantic wave of Lesson 1

The ‘Semantic wave’ is a way of representing the way in which meaning is communicated through time in relation to Semantic density and Semantic gravity (Macnaught, Maton, Martin and Matruglio, 2013). It is a particularly useful tool in this study because, the way in which the Semantic density and gravity developed temporally through a single lesson, explains something about Gillian’s use of representations in her teaching, which is an important aspect of understanding how Gillian communicates meaning in her classroom practice. I decided to include the analysis of one lesson only using the Semantic wave because analysis of both lessons showed a very similar picture. The one included here is another aspect of the translation device in this study – aiming to provide clarity with regards to my interpretation of the data in relation to LCT. In previous studies, the Semantic wave has been mapped as a single line, however, to show the nuance of Gillian’s practice, I have shown two separate lines - the dark grey showing how the Semantic gravity developed and the light grey showing the development of Semantic density throughout the lesson (figure 22).

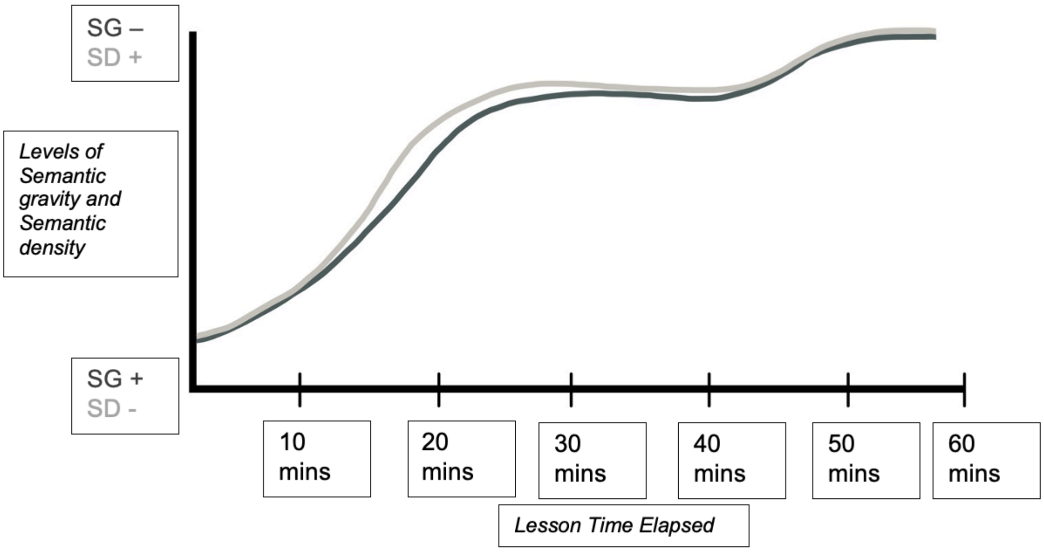


Figure 21 - The Semantic wave for lesson observation 1. The dark grey indicates the development of Semantic gravity and the light grey Semantic density.

Analysis of this lesson using the Semantic wave shows five key points that are important to this study. First, the beginning of this lesson shows Gillian starting with high levels of Semantic gravity and low levels of Semantic density, suggesting a 'prosaic' code (SG+, SD-). As the lesson progresses, there is a gradual shift away from context dependence and simplicity, towards the more abstract and complex, and after about twenty minutes Gillian's lesson would fall into a 'rhizomatic' code (SG-, SD+). Across the whole lesson, there is a general trend away from simple meanings and context dependence and towards complexity and abstraction. By the end of the lesson, all representations used to communicate meaning are abstract in nature and there are high levels of complexity. This was a very similar trend to the second observed lesson. Although there were minor differences in the timings of both lessons, each one began with high levels of Semantic gravity and then moved gradually away from this. This suggests that, in her teaching, Gillian deliberately started her lessons in a highly contextualised manner, before encouraging pupils to make generalisations, moving towards understanding the abstract nature of the lesson content. This is perhaps not surprising given the findings from the thematic analysis, especially the themes 'Mathematics for the people, by the people' and 'Using Representations for Mathematical Thinking', which showed how Gillian values contextualising mathematics in real-life for pupils, for them to develop deep understanding. An important difference between both lessons, however, was that the second lesson did not start with such low levels of Semantic density. This was due to the immediate use of abstract mathematical symbols in the second lesson. The textbook analysis showed that whilst the first observed lesson was right at the start of the fractions chapter, the second observed lesson was in the middle, and this explains why Gillian was able to start the lesson straight away with the abstract symbols. The pupils in Gillian's class had nine previous lessons to become comfortable and re-acquainted with these. The textbook analysis also showed this to be a general trend across this chapter – the Semantic density at the start of each textbook lesson increases, with the introduction of more complex representations and other mathematical ideas, such as measures.

The second important feature of the Semantic wave shown above is how the lines for both Semantic gravity and density stop increasing for a period around the mid-point of the lesson. At this point, the pupils were writing in their journals about what they had been doing and most of their responses were focused on the initial problem context. Gillian was encouraging them to think about the contextualised problem, but in terms of abstract fractions. Therefore, the Semantic gravity and density are still positive, but also somewhat tied to the beginning of the lesson as Gillian encourages them to think back. This slowing down of the trend towards complexity and context-independence seems to be an important part of the lesson, as Gillian is providing pupils time to reflect on what they have done already instead of pushing on towards potentially more tricky lesson content. Importantly, this trend was also found in the analysis of the second observed lesson.

The third important feature is the difference in the way Semantic density and Semantic gravity develop. In figure 22, the level of Semantic density rises before the level of Semantic gravity decreases. This shows how Gillian raised the level of complexity by introducing different representations (in particular, standard mathematical symbols) whilst keeping the content firmly contextualised. The moving away from the context only happens once the

Semantic density has already risen. This is a similar picture to the second observed lesson and suggests that Gillian's classroom practice promotes the development of complexity before the development of abstraction (SG-). It is also worth noting that, unlike with the Semantic gravity, the Semantic density does not drop back to negative levels at any point once it has risen. This is related to the way in which Gillian uses mathematical symbols. Once she has introduced these (thus raising the level of Semantic density), at no point in either lesson does she revert to not using them again.

The fourth important feature of this Semantic wave relates to the textbook analysis. When looking at the way the textbook content for this lesson develops with regards to Semantics, the development is somewhat like the real-life enactment that was Gillian's lesson. The textbook content starts off with a contextualised problem and then becomes increasingly abstract and complex, with the introduction of multiple representations. However, one important difference is that the textbook does not slow down the increase in Semantic gravity and density at the lesson mid-point, as Gillian did. This suggests that doing this is an example of Gillian exercising her own professional judgement about what her pupils needed at that moment in time. Her practice is not necessarily deviating from the textbook content, however, she is using it in a flexible way, considering what she believes the needs of her pupils are (in this case, to have more time reflecting on what they have done before increasing the complexity). Again, this is perhaps not surprising as it mirrors the findings within the thematic analysis theme 'Teacher and textbook collaborating', which showed how Gillian used the textbook in a confident way to ensure her lessons would meet the needs of pupils in her class.

Finally, an important feature of this lesson that is not so well captured within the Semantic wave diagram, is that Gillian would constantly refer to the simple initial problem context (a jam roll split into equal parts). Although the abstraction and complexity increase significantly as the lesson progresses (SG-, SD+), because Gillian continually draws pupils' attention back to the initial problem context, it is likely that the steep shift towards abstraction and complexity is tempered slightly and not felt in such an extreme way by the pupils as might be expected. This demonstrates a slight flaw with representing the lesson as a temporal wave in this way, because of Gillian's constant moving back and forth between complex and abstract, and contextualised and simple, it is hard to fully capture every element of how she uses representations to communicate mathematical meaning.

Summary of Analysis Applying the Semantic Dimension

As with the analysis using the specialization dimension, the previous examples are designed to provide a guide for the reader and to explain how each data segment has been translated into a code within the Semantic dimension. Figure 23 below illustrates how all the lesson segments from both observed lessons were coded. Although the previous examples provide important nuanced information about how Gillian communicated meaning in her lessons, it is also useful to see where all lesson segments lie within the Semantic dimension as this provides a fuller picture of her practice.

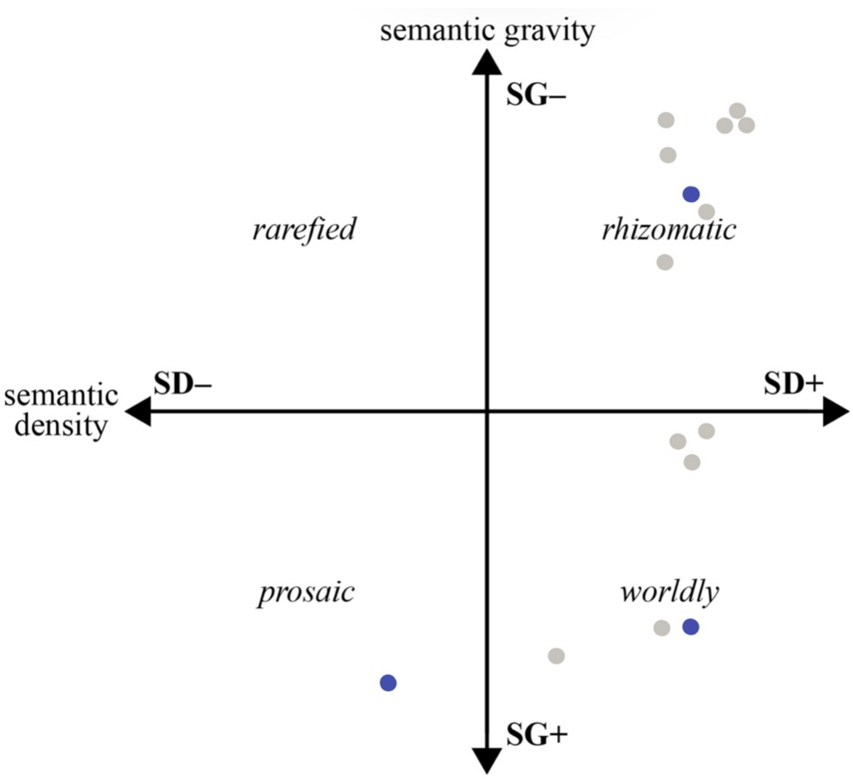


Figure 22 - A representation of all lesson segments plotted onto the Semantic dimension (exemplar segments in blue)

As figure 23 shows, most lesson segments fell into the ‘rhizomatic’ code, although there were also quite a few that were coded as ‘worldly’ and just one coded as ‘prosaic’. As the Semantic wave demonstrated, both lessons began with high levels of Semantic gravity (in either the ‘prosaic’ or ‘worldly’ quadrant) but then gradually progressed through ‘worldly’ and into the ‘rhizomatic’ quadrant. Therefore, the lesson segments in the ‘rhizomatic’ quadrant are all from the middle to the end of each lesson. This provides important insight into how Gillian uses representations to communicate meaning in her lessons. The end point, or perhaps the goal, of each lesson is to get to a point where mathematical meaning is generalised, de-contextualised and connections can be made so that pupils understand the complexity of the chosen mathematical content. Nevertheless, Gillian chooses to start each lesson off in a much more simplistic and contextualised manner. This is perhaps not surprising given that the textbook analysis showed that the textbook design for each lesson also follows a very similar pattern – beginning with contextualised problems and moving gradually away from these, and towards abstraction. Gillian sees contextualisation and simplicity as a means by which she can carefully guide the pupils into understanding highly complex and de-contextualised mathematical ideas. Within her practice, it seems that understanding mathematical meaning as complex, interconnected and abstract is the basis of success, but that the route to being able to do this begins with simpler and contextualised meanings.

An important finding within this phase of analysis was also that there were no lesson segments that were deemed to fit into the ‘rarefied’ code section of the Semantic dimension. Due to the complex and interconnected nature of mathematics as a subject discipline, it is difficult to consider an example of where school maths teaching might be considered a ‘rarefied’ code, where meaning is de-contextualised but also simplistic. Implications of this finding will be discussed in detail within the preceding chapter, but it is important enough to merit a brief mention here. In every single lesson segment, when Gillian moves away from context dependence, the Semantic density quickly becomes stronger (SD+). This is not surprising given the findings within the first phase of analysis in the theme ‘Using representations for mathematical thinking’, that showed how Gillian strongly emphasises making connections and mathematical thinking in her teaching.

6 Synthesis and Discussion

In this chapter I aim to bring together and synthesise the findings from each phase of analysis as well as position the findings in relation to the relevant literature. To ensure that there is clarity about how the findings from the three phases of analysis interact with one another, in each section I have first synthesised these three phases into one coherent summary. Each synthesis will serve to clarify and distil the key findings of my study. For each section, I will then present a discussion of how this summary relates to the literature. The synthesis presented in this chapter will answer the main research question through addressing the four areas related to the guiding questions, which cover the topics of: mathematical beliefs and knowledge, using representations, the influence of the textbook, and a summary outlining the relationship between all of these. This will then provide the essential foundations for the final concluding chapter (chapter 7) where the main research question will be answered.

To present this discussion with clarity, it is important to offer a reminder to the reader of the main aims and research questions. The primary aims of this study are first to contribute to the body of knowledge about teachers' use of representations in the school maths classroom and to focus upon teacher beliefs and knowledge as an influencing factor. Second, it aims to help teachers better understand beliefs about the nature of mathematics and how these might relate to the way in which knowledge is recontextualised into school maths. Finally, a third aim is to contribute more broadly to understanding the knowledge practices of teachers through critically applying LCT, and to use this to generalise to theory, extending the explanatory power beyond the specific context of the study itself. To achieve these aims, I seek to answer the question:

How do teachers' mathematical beliefs and knowledge influence their use of representations in the process of negotiating the mathematical meaning of fractions?

To fully answer this question, there are also four related questions that help focus the study. These questions are designed to direct the research process towards key areas that will help answer the over-arching research question above. These will be used to structure the ensuing chapter:

5. *How can we effectively understand teachers' beliefs and knowledge about the nature of mathematics and mathematics education?*
6. *How can we effectively understand how mathematical representations are used by teachers to communicate mathematical meaning in school maths lessons?*
7. *How does a textbook scheme influence teachers' beliefs and knowledge, and use of representations?*
8. *How can we explain the relationship between teacher beliefs and knowledge and the use of mathematical representations in the classroom?*

6.1 Mathematical Beliefs and Knowledge

In this section I will first draw together the findings related to mathematical beliefs and knowledge from all three phases of analysis (section 4.8). Following on from this I will critically discuss these in relation to the pertinent literature, answering the question, '*How can we effectively understand teachers' beliefs and knowledge about the nature of mathematics and mathematics education?'*.

6.1.1 Mathematical Beliefs and Knowledge: synthesis of findings

Throughout each phase of analysis, there are two key ideas that appear and relate to one another regarding the mathematical beliefs and knowledge of the case study teacher: a belief in the importance of specialised mathematical knowledge combined with a belief in the importance of social learning including learner dispositions.

The first idea was apparent throughout all phases of analysis and is about the central importance of specialised mathematical knowledge. Teachers having this knowledge and pupils developing it through teaching, formed a central part of Gillian's belief and knowledge system. There was a strong connection between her espoused beliefs in this and her actual classroom practice. In both observed lessons I identified there being a precise mathematical focus and in the ensuing interviews, Gillian confirmed this as a deliberate choice on her part by expressing a strong desire for pupils to learn specific mathematical content. Such a strong emphasis on precise mathematical knowledge outcomes for pupils demonstrates how Gillian firmly believes in its importance. The first phase of thematic analysis supported this within the theme 'Balancing pupil autonomy with teacher control' (section 5.5.3), where it was found that, whilst Gillian left a large amount of freedom for pupils to learn with autonomy in many areas, she also kept quite tight control over the presentation and sequencing of mathematical content in her lessons. Alongside this, within the theme 'Using representations for mathematical thinking' (section 5.5.6), it was clear that the type of representations, and the way in which they were used, was also something that Gillian controlled quite carefully, making decisions about what type of representation to use and for what purpose. The analysis using the specialization dimension from LCT (section 5.7.1) showed that many data segments (approximately half of them) demonstrated a positive epistemic relation, meaning that pupils demonstrating secure subject knowledge was a large part of the basis of success in her lessons. Supporting this, Gillian also espoused a belief that it is important for teachers to have strong subject knowledge about the mathematics they are teaching, so that they can adequately support pupils (section 5.5.3). This shows that, within Gillian's beliefs and knowledge system, there is a connection between teachers' subject knowledge and pupil learning, and that this is something she sees as important. For Gillian, the development of specialised mathematical knowledge both for teachers and pupils plays a central part in mathematics education and it is the teacher's job to manage this carefully.

Not only did Gillian believe this was important, but she also demonstrated secure specialised knowledge of fractions herself. The way Gillian responded to the teacher problem tasks, and the analysis of this using the data instruments (section 5.6.1), showed that she had what I describe as a ‘confident knowledge’ of fractions, including associated representations. This helps to explain why she feels confident in her belief about the central place of mathematical knowledge in school maths teaching. Because she is confident in her own knowledge, she feels secure in expressing this as being a key ingredient to successful school maths teaching. Overall, this shows that, in the case of Gillian, the development of mathematical subject knowledge is a very important part of school maths. This includes teachers being confident in their own knowledge, as well as pupils developing understanding of specialised mathematical knowledge as the basis of success in her lessons.

The second idea that relates to Gillian’s belief and knowledge system focuses on her belief in the importance of social learning and learner dispositions and may be seen in some ways as a tension with her firm emphasis on the centrality of mathematical knowledge. However, by applying the specialization dimension of LCT, it is possible to explain how these two beliefs co-existed and jointly formed an important aspect of Gillian’s belief and knowledge system. The analysis applying the specialization dimension (section 5.7.1) demonstrates that almost all the data segments could be interpreted as having a strong social relation, even when there was also a strong epistemic relation. This explains how, in Gillian’s classroom, the underpinning basis of success for pupils involved the development of social learning and learner dispositions alongside specialised mathematical knowledge; described in LCT as an ‘elite’ code. In particular, Gillian believed that it was important for pupils to become independent learners who were resilient, autonomous and would habitually communicate their thinking to one another and that it was the teacher’s job to promote these dispositions in lessons. For her, being a successful school mathematician involved these social relations as well as the development of specialised knowledge. Within the thematic analysis, in particular the theme ‘Mathematics for the people, by the people’ (section 5.5.1), it was clear that seeing mathematics as being there as a tool to be used and learned about socially, was of key importance to Gillian and something that she wanted her pupils to understand. For her, this is one of the main purposes of school maths.

This finding was reiterated in the analysis using the data instruments (section 5.6.1), specifically when looking at Gillian's global beliefs about mathematics where she demonstrated fallibilist beliefs about the nature of mathematics. However, the application of the data instrument showed a slight tension between these global beliefs and her more context specific beliefs, where the importance of acquiring specific knowledge, as dictated by the teacher and formal curriculum, became stronger and her belief in mathematics being a social tool was downplayed slightly. Nevertheless, the application of LCT shows how, rather than explaining this as a tension in her beliefs, it is possible to see these two aspects as combined to form her belief and knowledge system.

These two key ideas, the importance of mathematical knowledge, and the place of social learning including learner dispositions, are both highly important as they help explain Gillian's belief and knowledge system. They show that it is possible, in the case of Gillian, to balance these two ideas within school maths teaching, so that there is a focus on mathematical knowledge, alongside there being the development of social learning and learner dispositions. Applying LCT, this shows that a large amount of the data segments could be described using an 'elite' code, meaning that both social relations and specialised knowledge were the basis of success, rather than one or the other. It is likely that Gillian's 'confident knowledge' goes someway in helping her achieve this – being confident in the mathematics she is teaching is likely to leave her with a greater capacity to focus on the development of less knowledge focussed aspects of teaching, such as being resilient, autonomous, and communicating thinking. Having synthesised the three phases of analysis above in order to demonstrate the key ideas that relate to Gillian's belief and knowledge system, it is now important to discuss these findings in relation to the wider literature (sections 2.1 & 2.2).

6.1.2 Mathematical Beliefs and Knowledge: discussion

Within the literature, there is a significant amount written about teachers' beliefs and knowledge in relation to mathematics teaching (Fennema and Franke, 1992; Thompson, 1992; De Corte, Op't Eynde and Verschaffel, 2004; Hill, Schilling and Ball, 2004; Kuntze, 2021). This section will discuss the findings of my study in relation to this, specifically, covering the topics of availing beliefs and knowledge (section 2.2), beliefs about the nature of mathematics (section 2.1) and belief and knowledge systems (section 2.2.2 & 2.2.3).

Within the literature, it can be seen that there is a range of research that identifies both the types of beliefs that lead to improved pupil outcomes (Muis, 2004) and also the range of teacher knowledge that is seen as important in order to enable this, especially in relation to the teaching of fractions (Kieren, 1976; Fennema and Franke, 1992; Ma, 1999; Ball, Thames and Phelps, 2008; Hackenberg, 2013). Within my theoretical framework (section 3.3), I have adopted the term 'availing', as utilised by Muis (2004), to describe the type of beliefs and knowledge that are more likely to lead to desirable pupil outcomes, according to the literature. The choice of the term availing is different from others who describe certain beliefs as 'naïve' or 'sophisticated' (Schommer, 1990) and is designed to avoid any value judgements about specific beliefs. As has been shown in the findings chapter (section 5.6.1), Gillian had a confident knowledge of fractions and held beliefs about mathematics and mathematics teaching that were availing. This includes a detailed knowledge of the associated representations of fractions and a detailed understanding of how these can be used to communicate mathematical meaning. This is somewhat at odds with previous findings, which have shown fractions to be an aspect of mathematics that teachers are generally less confident with, even if they are a specialist, as Gillian is (Askew et al., 1997; Ma, 1999). An interesting aspect of this, that will be discussed later in this chapter (section 6.3), is the way in which the textbook supported this knowledge. In addition to understanding that Gillian had mathematical subject knowledge that was availing, it is also important to better understand her beliefs about the nature of mathematics in relation to the literature.

Ernest (1991) (section 2.1) considers beliefs about the nature of mathematics to fall on a spectrum between two main categories: fallibilism and absolutism. On one end of the spectrum, absolutist beliefs cover a range of philosophical viewpoints, which all focus upon the subject of mathematics as being a set of unchallengeable truths. In this view, mathematics is a pre-existing set of truths that are there to be discovered by people. On the other end of the spectrum, fallibilism covers a range of philosophical viewpoints which see the subject as created by human beings and thus fallible in nature (ibid., 1991; Hersh, 1999). Within the fallibilist belief category, Putnam (1975) describes beliefs about the nature of mathematics as 'quasi-empiricist', where there is a strong emphasis on the created nature of mathematics, and mathematical knowledge is developed through social interaction and critical debate. This is important as it is specifically this form of fallibilism that aligns with Gillian's beliefs. More in-depth discussion of these can be found in the literature review chapter (section 2.1). Within my study, Gillian's beliefs about the nature of mathematics were predominantly aligned with a fallibilist stance (section 5.6.1). This means that she emphasised mathematics as a subject where meaning was developed through social interaction and one that was designed to be used by people. Additionally, throughout the data corpus, she displayed a strong belief in mathematics being a subject where reasoning and justification of ideas played a central role. This suggests that her beliefs about the nature of mathematics were aligned with the 'quasi empiricist' approach, where the subject is seen as socially created and new ideas are accepted through a robust scientific approach (Polya, 1957; Putnam, 1975; Lakatos, 1976; Ernest, 1991) (section 2.1). In fact, Gillian's comment that 'maths is what you need it to be' [Gillian, interview 2], seems similar in nature to Lakatos' (1976: 146) idea that "Mathematical activity is human activity". Nevertheless, her fallibilist beliefs were not always so strong, and analysis of her beliefs in more context-specific situations showed that, at times, her beliefs became more focussed on pupils understanding specific mathematical rules or concepts as dictated by her as the teacher and the formal curriculum, rather than having the time and space to develop these ideas for themselves in a social manner. This is more aligned with some elements of an absolutist standpoint because mathematical knowledge is being presented as absolute fact by the teacher, rather than something for pupils to create for themselves through social interaction. This exposes a slight tension – at a global level, she wanted pupils to experience mathematics in the fallibilist, quasi empiricist way, however, she also had other external pressures, such as national tests, that meant she needed pupils to be able to enact certain mathematical activities within a given timeframe (by the time the tests were due to happen). Arguably, this is an example of Bernstein's (2000: 445) idea of the "discursive gap" between theory and empirical data that outlines how, often, empirical data cannot be fully explained by the theory used in a study and this is often due to inadequacies in the theory itself (section 3.2). In the case of my study, the theory within the literature (that beliefs fall into either fallibilist or absolutist categories) does not help explain the case of Gillian and her own beliefs about the nature of mathematics at all levels of globality (Törner, 2002). Two options would be to either say that she has different beliefs at different times, or that her beliefs are not congruent with her classroom practice, and this shows an inconsistency. However, this is not particularly useful when it comes to trying to understand how teachers' beliefs influence their practice and is similar to conclusions drawn by previous studies (Erikson, 1993; Raymond 1997; Philipp, 2007). Given that one of the aims of this study is to try and help teachers better understand how their beliefs might influence the recontextualisation of mathematics into school maths via their classroom practice, it is necessary to think differently and avoid the potentially false dichotomy of beliefs falling into one of two categories. This leads us to the idea of belief systems (Thompson, 1992; Leatham, 2006) or more importantly, as I have developed it within this study, belief and *knowledge* systems (section 2.2.2). This is because beliefs and knowledge can be seen as entangled (Kuntze, 2012; Dreher and Kuntze, 2015), and it is arguably more useful to study the two together so that the findings of my study can be better explained.

Within this study, I have analysed teacher beliefs and knowledge together, rather than separately, as has often been the case in mathematics research (sections 2.2.2 & 2.2.3). Nevertheless, the literature review conducted (chapter 2) demonstrated a lack of tried and tested theoretical frameworks that help facilitate this. Building on the suggestion by Thompson (1992) that beliefs exist within belief systems, I developed this idea further into 'belief and knowledge systems'. From here, I utilised Kuntze's (2012) model of teacher knowledge and beliefs, alongside the LCT dimension of Specialization (Maton, 2014) to create a theoretical framework (chapter 3) that allowed me to better explain Gillian's belief and knowledge system. Doing this enabled me to avoid having to explain away aspects of Gillian's beliefs and knowledge by describing them as inconsistencies, as others have done in the past (Philip, 2007), and instead provide a more detailed explanatory account of them.

Overall, her belief and knowledge system can best be summarised as a careful balancing act between the importance of mathematical knowledge, alongside the development of social learning and learner attributes in relation to mathematics. Using the lenses of Kuntze's (2012) model (see section 2.2.3 for further discussion), it is clear that Gillian has generally strong subject knowledge of fractions, or what I refer to as 'confident knowledge' to acknowledge that her belief is that her own knowledge is strong, and how to teach them in all four of Schulman's (1986) categories (pedagogical knowledge, pedagogical content knowledge, curricular knowledge and subject matter knowledge) (section 2.2.3). In addition to this, at a global level, she has a belief in, and knowledge that, mathematics is a subject that is there for society to use, and that social learning of the subject is important. As discussed previously, this seems to align with the quasi-empiricist stance (Polya, 1957; Putnam, 1975; Lakatos, 1976; Ernest, 1991), which falls within the fallibilist category of beliefs. However, when we look at her beliefs in a more context specific situation, at times they seem to become less focussed on social learning and more focussed on acquisition of specific knowledge, which may seem to be less fallibilist in nature. Therefore, it is arguable that using the terms 'fallibilist' or 'absolutist' to categorise beliefs is not helpful and, instead, using a different approach to belief and knowledge systems is likely to yield more detailed explanatory findings. I argue here that the application of LCT to help explain a belief and knowledge system is more useful for teachers and helps connect the empirical data to theory in a more precise way. Within the case of Gillian, almost all the data segments fell into either a 'knower' code or an 'elite' code (section 5.7.1). This means that almost all the time, she was promoting the development of social learning and learner attributes (referred to in LCT as 'social relations') as the basis of success in her lessons. She believed, and knew, that if pupils were to develop these things, then they would be more likely to be successful in school maths. Additionally, for approximately half of the data segments, she also strongly promoted the acquisition and use of specialised mathematical knowledge (referred to in LCT as ‘epistemic relations’) as the basis of success alongside these social relations. Importantly, the extent to which one or the other (or both) were promoted as the basis of success, depended on her perception of the needs of her pupils. This shows that Gillian’s belief and knowledge system was one where she valued both social relations and epistemic relations and the extent to which one took over more strongly than the other was very much based upon her assessment of what was needed at any one time. Additionally, the fact that she had a confident knowledge of fractions is likely to have supported this. The overall picture here is of a belief and knowledge system that is fluid in nature and adaptable to a variety of social situations. It is arguable that in mathematics, both effective learner dispositions and the acquisition of knowledge are of key importance to success. For example, the literature around mindset in mathematics shows that the development of a positive mindset towards the subject, one aspect of learner dispositions, can lead to better learning outcomes for pupils (Sun, 2015, Boaler, 2016). Alongside this, it is widely recognised within the literature that for pupils to succeed in mathematics, they do need to acquire specialised mathematical knowledge, including the use of multiple representations (Duval, 2006; Ball, Thames and Phelps, 2008; Carbonneau, Marley and Selig, 2013; Dreher and Kuntze, 2015). This would mean that teacher belief and knowledge systems that fall predominantly into an ‘elite’ code within LCT, are likely to lead to better outcomes for pupils. In this way, I argue that the application of LCT to the study of belief and knowledge systems contributes a new perspective on the issue - one that may serve to be of greater practical use for teachers than previous research. Nevertheless, it is now important to consider how Gillian used representations in her teaching before moving on and comparing how her belief and knowledge system might have influenced this.

6.2 Using Representations

As with the previous section, prior to discussing findings in relation to the literature, this section will first draw together the findings related to Gillian's use of representations from all three phases of analysis (section 4.8), answering the question, '*How can we effectively understand how mathematical representations are used by teachers to communicate mathematical meaning in school maths lessons?'.*

6.2.1 Using Representations: synthesis

Building on from the notion of Gillian's 'confident knowledge' in the previous section, it appears that the knowledge she possessed translated into a confident and careful use of representations in her lessons. Throughout the data corpus there were several key themes that arose, however there are two ideas that were pertinent to all three phases of analysis: dialogue and representations, and the purpose of representations. These are particularly important ideas to my study because, between them, they provide a summary of how Gillian used representations to teach fractions, thus helping answer the research question.

The first idea is that Gillian both believed, and practised, a fervent use of classroom dialogue alongside the use of multiple representations in her lessons. This is best captured in the theme 'Conversations and representations to understand mathematics' (section 5.5.5) but was also apparent across all phases of analysis. The key point here is that Gillian used dialogue, both as a whole class and amongst small groups of pupils, as a way of facilitating the communication of mathematical meaning alongside the use of multiple representations. However, it was not the case that there was the same amount of time given for talk at every stage in the lesson. The lesson observations (section 5.3) showed that there was a predominance of dialogue at the beginning and that this reduced as each lesson progressed. Additionally, the analysis using the Semantic dimension of LCT (section 5.7.2) demonstrated that representations used at the start of lessons tended to be coded as more ‘worldly’ in nature, meaning that they were more strongly contextualised but still had relatively dense meaning. At these times in both lessons, it was also the case that the widest range of representations were being used – two or more in each instance. This suggests that Gillian’s use of dialogue was focussed upon enabling pupils to communicate and develop mathematical meaning by drawing connections and comparisons across multiple representations. This also supports Gillian’s belief in the importance of specialised mathematical knowledge discussed previously (section 6.1.1), as it highlights how her use of representations was a tool she could use to help pupils develop this knowledge for themselves. As each lesson progressed, the analysis using the Semantic dimension showed that representations became less contextualised, whilst remaining dense in meaning, and thus were coded as ‘rhizomatic’. In practice, this meant that the representations used became more abstract in nature, with an emphasis on formal mathematical symbols. As this happened in each lesson, there tended to be less time given for dialogue, especially between pupils in small groups. This suggests that Gillian saw the more contextualised use of multiple representations, that she used at the start of each lesson, as more relevant for discussion and that there was an expectation that pupils would be able to generalise the mathematical meaning from these and use less context specific representations, such as abstract symbols, during independent practice where less time was given for dialogue. It is important to point out here however, that at no point was there no dialogue, so, even towards the end of lessons where time for dialogue was reduced, it was still happening.

The second idea is that Gillian tended to use representations as a stimulus for mathematical thinking. The whole point of using representations for her was to enable a deeper communication of meaning in her lessons. On a surface level, it was apparent that she used *multiple* representations in each of her lessons and this was something that she advocated as being important in the interviews. She did this because she believed that if pupils could use multiple representations flexibly, both with and without real-life contexts, they would gain a deeper understanding of the mathematics she was teaching. Drawing together all three phases of analysis, there are two important points to make here – one about the ‘end goal’, or basis of success in her lessons, and one about the way in which representations were used to help pupils get there. Because Gillian used multiple representations at the same time in her lessons, I interpreted almost every lesson segment to have a semantically dense level of meaning being communicated. This means that the meaning being communicated with the representations was complex and interconnected in nature. The main thing in her lessons that shifted, was how context dependent (level of semantic gravity) the representations used were. For example, in one lesson she moved from using representations of jam rolls split into equal parts as part of a ‘real-life’ problem, to using rectangular area models split into equal parts where there was no real-life context given, however both representations were used alongside symbolic representations of the fractions. My analysis showed that the ‘end goal’ of each lesson was that pupils would be able to complete mathematical problems that were predominantly context-independent yet complex in meaning, which would fall into a ‘rhizomatic’ code within the semantic dimension of LCT. This usually involved completing fraction problems with abstract mathematical symbols, sometimes alongside area model representations. This means that the basis of success, or the ’end goal’, in Gillian’s maths lessons was that pupils would be able to use multiple-representations to solve problems that were not dependent on any context (this does not mean that they would not be able to solve context-dependent problems, however). This is something that was re-iterated in all three phases of analysis. Nevertheless, it was also clear that to help pupils get to this ‘end goal’, Gillian scaffolded pupil learning by using context-dependent examples, which were coded as ‘worldly’. This suggests that representations that could be coded as ‘worldly’ were being used by Gillian as a way of scaffolding the communication of mathematical meaning, so that eventually pupils can work with examples that fall into a ‘rhizomatic’ code. Related to this, Gillian did also use representations to help pupils get into a more positive mindset about fractions: this was important because she believed that the pupils needed to be in a positive mindset before they could engage in deep mathematical thinking. Therefore, using representations to influence pupils’ affect towards mathematical ideas was also for the purpose of developing mathematical thinking.

Both key ideas (dialogue and representations, and purposeful use of representations) are important findings within this study and help contribute towards answering the research question because they provide insight into Gillian's practice involving representations. Specifically, they demonstrate how the application of LCT as part of my theoretical framework can be used to explain the complex way in which multiple representations might be used to communicate mathematical meaning. It is now important to discuss the findings related to use of representations in relation to the literature, as discussed in the literature review chapter (sections 2.1 & 2.2).

6.2.2 Using Representations: discussion

Within the literature, it is widely agreed that using multiple representations to teach school maths is a highly important and effective practice (Goldin and Shteingold, 2001; Duval, 2006; Carbonneau, Marley and Selig, 2013). This is particularly the case with fractions, which can be seen as a multi-faceted concept with multiple meanings and associated representations (Charalambous and Pitta-Pantazi, 2007; Dreher and Kuntze, 2015; Gabriel et al., 2013; Hackenberg, 2013; Panaoura et al., 2009; Tunç-Pekkan, 2015; Rau and Matthews, 2017). This section will report on the findings of this study in tandem with the relevant literature, specifically focussing on two areas: use of representations for teaching (section 2.3), and types of representations used (section 2.3.2). Specific reference will also be made to representations in the domain of fractions (section 2.3.4) throughout.

First, it has already been established that Gillian used *multiple* representations in her teaching (section 5.6.2), and this is acknowledged in the literature as an effective teaching practice (Goldin and Shteingold, 2001; Duval, 2006; Carbonneau, Marley and Selig, 2013). Despite this, it is the way in which multiple representations are used that is the key to making sure any potential gains in learning are capitalised upon (Goldin and Shteingold, 2001; Duval, 2006; Rau et al., 2009; Carbonneau, Marley and Selig, 2013). Duval's (2006:

111) concept of representational registers suggests that it is important for pupils to learn how to use individual types of representations, what he refers to as "treatments", whilst also being able to see the connections between different representations, what he refers to as "conversions". Within my analysis, it was clear that both types of representational activity were happening in observed lessons. In her teaching Gillian spent time comparing and looking for connections between different representations and emphasised pupils' ability to see these connections (the ability to make conversions) as a basis for success in school maths. Additionally, she focussed in specifically on certain representations as ones that she felt pupils needed to master the treatment of, most notably rectangular area models and abstract mathematical symbols. This suggests that conversion between different representations was important but also, that certain representations were valued more highly than others as worth spending time on. This practice mirrors the research generated by the Rational Number Project (RNP), which demonstrated how pupils could develop deeper understanding of fractions when teaching involved multiple representations and focussed upon teaching pupils to make translations between them (Cramer, Post and delMas, 2002). This also shows how using representations in a semantically dense way and moving between high and low levels of semantic gravity leads to what the literature suggests is an effective use of representations. Therefore, it is arguable that seeing a 'rhizomatic' use of representations as the end goal, as Gillian does, and moving from a 'worldly' use of them into this, is more likely to lead to an effective use of representations in the process of communicating mathematical meaning.

As discussed in the previous section, an important part of how Gillian facilitated this type of representational activity was her use of dialogue. This is something that appears often within the literature as being of key importance to successful use of representations (Cobb, 2000; Sfard, 2000; Rau et al., 2009). In particular, speaking to the ontological nature of mathematical objects and the way in which mathematical meaning is developed, it is argued that the meaning of mathematical objects gains collective understanding through social discourse (Cobb, Yackel and Wood, 1992; Radford, 2006). The findings of this study show that Gillian's use of dialogue, or 'conversations' in her own terminology, was a key strategy that she used to help negotiate the mathematical meaning of fractions in her lessons. Arguably, this is one way in which her practice considered common criticisms of the internal/external view of representation (as discussed in section 2.3.3); by deliberately facilitating extended discussions about representations and their meanings as perceived by the pupils. I am not suggesting that these arguments are something that she explicitly knew about, however it does seem to be the case that the way in which she used representations in her teaching mirrors common social constructivist viewpoints (Cobb, Yackel and Wood, 1992; Cobb, 2000; Radford, 2006). Within the literature, such viewpoints argue that it is not just the use of mathematical representations alone that is important, but also the way in which they are used, referred to as "representational activities" (Sfard and Thompson, 1994: 2). Specifically, there is the suggestion that pupils need to be provided with opportunities to co-construct the meaning of representations through social discourse in order to arrive at a shared understanding of the mathematical objects they represent (Cobb, Yackel and Wood, 1992; Cobb, 2000; Radford, 2006). Gillian's constant use of dialogue about representations appears to mirror this stance. With specific reference to fractions, within the literature it has been shown that prompting pupils to "self-explain" when using multiple representations to solve problems can enhance learning outcomes (Rau et al., 2009: 442). Within their study they refer to 'self-explaining' as explicit time given for pupils to verbally reason about their use of fraction representations (ibid., 2009). The way in which Gillian facilitated dialogue in her lessons is very much aligned with this, suggesting that her practice is likely to lead to desirable learning outcomes. Not only this, but she also provided time for pupils to represent the mathematics for themselves, in informal 'jotters' and also more formal 'maths journals' (sections 5.4.3 and 5.4.6). This is another example of the way in which she used communication to facilitate the negotiation of meaning with representations (this time non-verbal and in the form of writing and drawings). This mirrors the research conducted by Meira (1995), which showed the importance of pupils being given time to create their own representations and argued that, in doing this, pupils were more likely to develop a fuller conception of mathematical objects. Nevertheless, the findings of this study did show that most pupils' own recorded representations were the same in variety as those that the teacher was using, or that were in the textbook. Therefore, it is not possible to determine the extent to which pupils were generating their own representations, or whether they were simply copying external representations that had already been shown to them. This draws attention to the specific type of representations that Gillian was using and how these compare to the types of representation discussed within the literature.

Within her lessons, Gillian predominantly used area model diagrams (both circular and rectangular), abstract mathematical symbols and verbal language to represent fractions. One of the predominant approaches to classifying representational registers might be considered the 'Concrete-Pictorial-Abstract' (CPA) approach (Merttens, 2012), which builds upon the work of Bruner (1966). Although the literature review chapter highlighted the problematic nature of this approach (section 2.3.2), specifically in interpreting what is meant by a 'concrete' representation, it is often argued in general terms that effective teaching utilises all three types of representation (Drury, 2018). Additionally, there has been particular focus on physical manipulatives as being an effective tool for computational fluency (Carbonneau, Marley and Selig, 2013). Within Gillian's teaching, it was clear that both 'pictorial' and 'abstract' representations were used throughout, however there was only one instance where something akin to a manipulative was being used. This was where she used strips of paper and asked pupils to represent a cake chopped into equal parts by folding it (section 5.5.2). Interestingly, this is one of the observations where I believed there to be less focus on the negotiation of mathematical meaning, and Gillian backs this interpretation up by pointing out that she spent time doing this predominantly to put pupils at ease with the idea of working with fractions. For her, the use of a manipulative in this instance was for broader purposes than just mathematical thinking and reflects Goldin's (2002b) assertion that representation as a concept includes a person's affect towards a mathematical idea. Here, I argue that application of the CPA approach to classifying representations is too broad and misses some of the complexities relating to specific mathematical content. Within the literature about fraction representations, more attention is given to the use of varied visual images than anything else (Rau et al., 2009; Prediger, 2011; Tunç-Pekkan, 2015; Rau and Matthews, 2017), as opposed to differentiating between physical manipulatives and visual images. In fact, according to one study (Tunç-Pekkan, 2015), pupils are more likely to find initial learning about fraction ideas easier with circular and rectangular area models, and this is exactly what Gillian did in the observed lessons, before moving on to making a connection with abstract symbols. However, it is worth noting that the representations Gillian used did not reflect the wide variety of fraction sub-constructs (Kieren, 1976) as is outlined in detail within the literature review (section 2.3.4). Nevertheless, the textbook analysis (section 5.4) did show that these would have been covered if her teaching had been looked at across the year, suggesting that the textbook influenced the rate and progression of how these were introduced (section 5.5.2) but also that teachers would have to carefully notice when these were introduced in other topics.

In summary, I describe Gillian's use of representations as being aligned with a social constructivist stance (section 2.3.3), where dialogue and non-verbal communication (such as writing and drawing) are used to negotiate the mathematical meaning of fractions. Gillian dedicates time in her lessons to allow pupils to co-construct the mathematical meaning of representations and how they relate to abstract mathematical objects. The way in which she uses dialogue seems to support the idea that mathematical objects are treated as having an ontological status that is developed through social discourse; in her classroom mathematical meaning is developed through dialogue and therefore mathematical objects are seen as ideas developed socially, rather than existing as absolute ideas to be discovered (Sfard, 2000). This supports the conclusions drawn in the previous section about her beliefs and knowledge in relation to fallibilist beliefs about mathematics, and mathematical meaning being developed through social learning (section 6.1). Perhaps one reason that Gillian found the time within lessons to do this was that she only used a small variety of representations. For her, choosing to use a carefully selected few representations and to have time to discuss these was more important that introducing a wider range of representations and not having the same amount of time for dialogue. This is related to the influence of the textbook, which will be further discussed in the next section. Additionally, a key contribution of my study is the way in which LCT has been used to explain the representational activity in Gillian's classroom. It is arguable that application of the 'CPA' approach to classifying representations is too broad to be useful for teachers when it comes to specific mathematical content, such as fractions, and that the application of the semantic dimension of LCT presents a more useful and more detailed way of explaining the use of representations in a progressive way. Specifically, moving from using representations that can be interpreted as 'worldly' through to a more 'rhizomatic' code, within and across a series of lessons, presents a new way of thinking about representational activity for teachers. It is possible that in some cases, where learners are truly novice to the concept being taught, that it may be worth starting with representations where meaning is not dense at all and very contextualised, which would represent a 'prosaic' code, however this is outside the scope of this study. This presents a new opportunity for research into this area and further work would need to be carried out to analyse whether this approach to teaching would lead to desirable outcomes for pupils.

6.3 The Influence of the Textbook

As with the previous sections in this chapter, the findings from all phases of analysis relating to textbook use will first be synthesized prior to a discussion in relation to the pertinent literature, answering the question, '*How does a textbook scheme influence teachers' beliefs and knowledge, and use of representations?'*.

6.3.1 The Influence of the Textbook: Synthesis

Across all phases of analysis, it was clear that the textbook that Gillian was using played an important role. This section will first explain the way in which the textbook influenced her beliefs and knowledge system and will then explain how it influenced her use of representations. In doing so, I will draw upon the previous two sections in this chapter.

It was apparent across all three phases of analysis that the textbook was an influencing factor upon Gillian's beliefs and knowledge. The textbook analysis showed that, within each textbook lesson, there were two important things that aligned with Gillian's belief and knowledge system. First, as is described previously (section 5.4), the textbook is split up into discrete 'lessons' and each lesson has a very precise learning objective that is stated within the teacher guidance and is designed to build progressively upon previous lessons and chapters. Within each of these lessons, the representations and problems given are designed to direct attention onto this specific mathematical objective. As is explained previously in this chapter (section 6.1.1), a key part of Gillian's belief and knowledge system was that the place of mathematical knowledge was important and developing deep understanding of the mathematics she was teaching was part of the basis of success for pupils in her lessons. This demonstrates an important congruence between the textbook and Gillian's beliefs and knowledge – both value an emphasis on pupils developing mathematical knowledge that is dictated from an external source (both the teacher and the textbook). This is also supported by the application of the specialization dimension of LCT, which showed that much of the analysis demonstrated that Gillian believed that developing specialised mathematical knowledge was a significant part of the basis of success in school maths. The analysis using the data instruments (section 5.6) also supports this as the one representation of fractions that Gillian was not secure on (iterating fractions – see section 5.6.1) was one that was not at all apparent within the textbook she was using (section 5.4). This suggests that her knowledge of representations had been developed by her use of the textbook. Additionally, throughout the interviews, Gillian expressed a high level of confidence in the textbook, suggesting that she would use examples from it even if she did not know why they were there, or did not agree with them fully. Within the findings chapter (section 5.5.4), I describe this as Gillian collaborating with the textbook to facilitate a precise and explicit focus on mathematical knowledge. Nevertheless, the second aspect of the textbook that mirrors Gillian's belief and knowledge system is that it also promotes a focus on pupil reasoning and justification of ideas (section 5.4). The textbook analysis showed that, within every discrete lesson, there were child-like cartoon characters who offered their own ideas, prompts and suggestions, inviting the reader to question and think critically about problems and representations. It is as if the characters in the textbook are designed to mirror an approach to mathematical learning that is social and involves critical dialogue (section 5.4). Instead of presenting one unified 'voice' of mathematics, the textbook presents ideas as coming from multiple perspectives (the cartoon characters) and thus suggests to the reader that mathematical meaning is socially constructed. This mirrors another aspect of Gillian's belief and knowledge system that is the emphasis on the development of social learning including learner dispositions, in this instance verbal reasoning and justification. This demonstrates another congruence between the textbook and Gillian's belief and knowledge system; one which is also supported by the LCT specialization dimension findings that show how, often, the basis of success in Gillian's teaching also involved the development of key social relations, alongside specialised mathematical knowledge.

As well as the textbook being an influence on Gillian's belief and knowledge system, it was also apparent that it influenced her use of representations. Mainly, this was related to the types of representations used, and the way in which they progressed within lessons. First, within the first phase of thematic analysis (section 5.5), it was clear that Gillian valued the use of a wide range of representations to teach fractions. During the belief and knowledge tasks interview, she suggested that she would use all the representations shown (section 5.5.4), however in the observed lessons she only used a small variety (circular and rectangular area models, abstract symbols, and spoken language). Importantly, the representations she used were also those emphasised most prominently within the textbook (section 5.4). This suggests that the given representations in the textbook caused Gillian to temper what she used within lessons. Second, the analysis using the Semantic dimension of LCT showed how the way Gillian's use of representations, moving from a 'worldly' code through to a 'rhizomatic' code in each lesson, mirrored the progression of representations as given in the textbook. The main difference was that Gillian controlled the amount of time spent discussing each part of the lesson, facilitating a greater emphasis on the initial, more contextualized problems. This means that she controlled the pace at which the lesson moved from ‘worldly’ to ‘rhizomatic’. Both ideas discussed in this section show how the textbook played an important role in shaping the way Gillian thought and acted with regards to her maths lessons. It is now important to consider the extent to which this reflects the pertinent literature so that these ideas can be situated within the broader research context.

6.3.2 The Influence of the Textbook: Discussion

Within the literature, it is recognised that school maths textbooks play an important role in helping determine the types of mathematical experiences for pupils, including exposure to different representations (Charalambous et al., 2010; Wijaya, Heuvel-Panhuizen and Doorman, 2015). Additionally, the specific cultural context of my case study means that many schools across England have been introducing government approved maths textbooks over the past six years. Perhaps the most obvious thing to draw attention to first is that Gillian used her textbook as the main source of mathematical content for her lessons and followed it carefully. She did acknowledge that there may be times when she would shift the order of lessons, however she still was using the textbook content instead of any other resource. This is at odds with the suggestion within the literature that, in England, teachers only used textbooks as the basis of their planning 10% of the time (Mullis et al., 2012). Bearing in mind the difference in date between the aforementioned literature and the subsequent introduction of government approved textbooks in 2016, it is possible to argue that the way in which textbooks are commonly used may have shifted within England. This is outside the scope of this study, but certainly merits further research, which some are currently undertaking (Barclay, Barnes and Marks, 2022). The textbook which Gillian was using was one that covered the statutory English National Curriculum (DfE, 2013a) content. However, it goes beyond this by splitting this mathematical content into discrete lessons within which specific representations are used to communicate meaning. In this way, this textbook was “selecting an offer of meaning” of the statutory curriculum content (Lilliedahl, 2015: 41). This would suggest that, on one level, the textbook was an example of the “planned curriculum” as it offers an interpretation of statutory curriculum content, set out in a series of progressive lessons and chapters and is also nationally recommended by the government (Gehrke, Knapp and Sirotnik, 1992: 55). In this case, the textbook is an example of where the subject discipline of mathematics has been recontextualised into school maths as a formal document (the textbook) and is therefore an element of the “official recontextualising field” (ORF) (Bernstein, 2000: 33). Despite this, upon further inspection, when looking specifically at how the textbook is influencing the communication of mathematical meaning in Gillian’s practice, it seems that it is more than just part of the ORF. As the previous section has demonstrated, there is an important congruence between the way in which the textbook presents mathematics as school maths, and Gillian’s beliefs and knowledge system as well as her actual classroom practice. It is not possible to separate the two and say that one has come before the other, rather it seems that the textbook and Gillian are both entangled in the process of negotiating the meaning of fractions within her lessons. Therefore, the textbook exists partly within the ORF, but also partly in the “pedagogic recontextualizing field” (PRF) alongside the teacher (Bernstein, 2000: 33). Rather than the textbook being a tool that can be described as “curriculum use” (Remillard, 2005: 212), it is more complex than this and Hetherington and Wegerif’s (2018: 27) concept of “material dialogic pedagogy” presents a more accurate way of explaining the role of the textbook within my case study. The way in which the mathematical meaning of fractions is negotiated with this case study is complex and involves interplay between Gillian’s beliefs and knowledge system, classroom practice where dialogue is central, and the textbook. Considering Barad’s (2007) theory of agential realism, in this way the textbook as physical ‘matter’ is an active part of the process of meaning making and cannot be separated from Gillian or the pupils in her class. The role of the textbook has not been the predominant focus of my study. However, the findings show that it is highly pertinent to the research question and its part in the negotiation of mathematical meaning cannot be ignored. It is now important to draw together the previous three sections and consider the relationship between mathematical beliefs and knowledge, and use of representations in the classroom.

6.4 Chapter Summary: The relationship between beliefs and knowledge, and representation use

This section draws together the previous three sections and answers the question, '*How can we explain the relationship between teacher beliefs and knowledge and the use of mathematical representations in the classroom?*', specifically in the case of Gillian. In doing so, the previous synthesis of findings, alongside the relevant literature will be discussed. I will first theorise about Gillian's belief and knowledge system and, second, her use of representations. Finally, this will be followed by a discussion of the relationship between the two.

First, Gillian's belief and knowledge system can be explained with four key ideas; a confident knowledge of fractions, a belief in the importance of mathematical knowledge, a belief in the importance of social learning including learner dispositions, and a belief in the nature of mathematics being socially created and for social use. These ideas have been discussed in detail within this chapter (section 6.1), however an important point to make is that, in the case of Gillian, it does seem possible that "knowledge blindness" (Maton, 2014: 4) can be avoided whilst promoting less subject-specific learner dispositions, such as resilience. This means that Gillian maintained a precise focus on the development of mathematical knowledge, whilst at the same time promoting other things in her teaching such as social learning, resilience and verbal reasoning. The application of the specialization dimension of LCT has helped explain how this is possible, by demonstrating the way in which Gillian both promoted epistemic and social relations within her beliefs and knowledge system, as well as classroom practice. It has demonstrated that the basis of success in Gillian's lessons was a careful combination of developing social learning and learner dispositions, as well as specialised mathematical knowledge. Not only this, but where previous research has found there to be so called inconsistencies with teacher beliefs and practice (Raymond, 1997), LCT has helped show that, by thinking about beliefs and knowledge as a system, it is possible for seemingly differing beliefs to be held and co-exist alongside one another. For example, Gillian believed that it was important for pupils to understand specific mathematical knowledge and that this came from her own expertise as well as the textbook she used and was something she kept relatively tight control of. Nevertheless, she also believed that pupils should develop independence in their learning and be able to act with autonomy. By applying the specialization dimension of LCT, it is possible to see how this can work as a belief and knowledge system rather than explaining it away as an inconsistency.

Second, Gillian's use of representations can be explained with three key ideas: confident and careful use; the central role of dialogue; and the progression of representations. These have been discussed in detail earlier on in this chapter (section 6.2). Importantly, with reference to the analysis using the data instruments (section 5.6), her use of representation can be seen as availing in nature, meaning that the way in which she used representations is likely to lead to desirable pupil outcomes, according to previous research. Nevertheless, the important point to make here is about the contribution to knowledge that this study makes through the application of the Semantic dimension of LCT. Previous research has shown that mathematical representations are not a straightforward area of study and there are ontological questions about the nature of mathematical objects and thus, what is being represented in any social situation is contested (Cobb, 2000; Sfard, 2000; Duval, 2006; Radford, 2006; Iori, 2017). Although it is widely accepted that using multiple representations in teaching will lead to desirable outcomes for pupils (Goldin and Shteingold, 2001; Duval, 2006; Rau and Matthews, 2017), it is still an area of study that needs greater focus to help teachers know what effective use of representations looks like. In this case study I apply the Semantic dimension of LCT to help explain the underlying structuring principles of Gillian's representation use and, in doing so, offer a new way of explaining representational activity in the classroom. I have argued that the commonly used 'CPA' approach (Merrtens, 2012) to describing types of representations lacks the necessary detail that will help teachers fully understand the complexity of how representations can be used to negotiate mathematical meaning. Although it provides a useful starting point in identifying more or less ‘concrete’ or ‘abstract’ representations, it does not consider the complexity of representations, described as semantic density within LCT. For example, it is possible to have a ‘concrete’ representation that is very complex in nature, as was seen in some of Gillian’s teaching where she used images of several sliced pizzas combined with abstract symbols to teach the subtraction of fractions. It would also be possible to have very simplistic ‘concrete’ representations, such as one pizza chopped into four equal parts to show quarters. Both would fall into the ‘concrete’ phase of the ‘CPA’ approach, but one is much more complex than the other. Additionally, the ‘CPA’ theory presents three discrete modes of representation (concrete, pictorial or abstract) whereas within LCT, it is recognised that through using combinations of representations alongside dialogue, the negotiation of meaning using representations is more nuanced and less disconnected in nature.

Finally, it is important to consider how Gillian’s beliefs and knowledge influenced her use of representations. Having a belief and knowledge system where mathematical knowledge, alongside social learning including learner dispositions, are both held as important has influenced the way in which Gillian uses representations. Her lessons demonstrate how she builds a culture of dialogue and autonomy whilst using multiple representations to effectively focus attention upon precise mathematical knowledge. Within her knowledge and belief system I describe her as having ‘confident knowledge’ of fractions and this translates into classroom practice where she uses multiple representations for fractions in a careful and confident manner. However, it is not just her beliefs and knowledge influencing this, as the textbook has been shown to also influence her practice. It is likely that her confidence with representations partly stems from her confidence in the design of the textbook she uses, as well as her own beliefs and knowledge. By using the textbook, Gillian is bringing another voice of mathematics into the process of meaning making within her lessons; that of the textbook authors and designers, and this is likely to have contributed to the way in which representations were treated. This supports the idea discussed previously (section 2.4) that the textbook simultaneously is part of the “official recontextualising field” (ORF) as well as the “pedagogic recontextualising field” (PRF) (Bernstein, 2000: 33). This means that, although the textbook is an official document that translates the National Curriculum (DfE, 2013a) content into an official book of school maths lesson content (ORF), it is also acting as a vehicle to recontextualise the formal curriculum content, which is then experienced by the teacher and pupils (PRF). Alongside this, the findings of my study also align with Hetherington and Wegerif’s (2018: 27) “material dialogic pedagogy” and Barad’s (2007) theory of agential realism (section 2.4), in that the textbook can be seen as an active part of the creation of meaning in Gillian’s lessons. The textbook and its associated representations come together with Gillian and her pupils to engage in an entangled discourse, where mathematical meaning is developed. Additionally, Gillian’s belief and knowledge system emphasised the importance of social learning and learner dispositions and this appears to be reflected in another way in which representations were used. Whilst using multiple representations in her lessons, Gillian also promoted these as objects for discussion and part of the basis of success in her lessons was pupils being able to communicate about representations to one another effectively. This demonstrates a high level of congruence between what Gillian espoused as beliefs and knowledge relating to social learning, and her actual use of representations where dialogue was central.

7 Conclusions

7.1 Introduction

My research has focussed upon the beliefs, knowledge, and practices of teachers and how these influence the communication of mathematical meaning in the primary school classroom. By conducting an in-depth case study of one teacher's beliefs, knowledge, and practices, I have been able to contribute new knowledge to the domain of mathematics education in five ways, which together form a working hypothesis. First, I have demonstrated the idea of a 'belief and knowledge system', within which beliefs and knowledge are integrated and by being considered in that way lead us to a better understanding of how they come together to influence practice. Second, by applying Legitimation Code Theory (LCT) as an analytical tool, I have been able to explain how it is possible for teachers to have a belief and knowledge system where both the acquisition of specialised mathematical knowledge and the development of social learning and learner dispositions are held to be important. I describe this as a careful balancing act between the two. Third, applying LCT, I have been able to provide new insight into how a teacher is able to use representational activity in the maths classroom, revealing how they move between more or less contextual and complex representations as a lesson progresses. Fourth, I have also demonstrated an example of how a textbook can be integrated with a teacher's belief and knowledge system, coming together with the teacher to communicate mathematical meaning within the classroom through its own voice. Finally, in addition to these, this study has made a methodological contribution by developing an innovative approach to analysis of teacher beliefs and knowledge and classroom practice. The three-phase hybrid thematic analysis, using the critical realist concept of *retroduction*, offers a new powerful approach to research in this area. In this chapter I will first explain how my study has answered the research question and then go on to summarise these contributions. This will be followed by a discussion of the recommendations and implications for policy and practice and then I will offer some insight into the limitations of my study and how these might help suggest opportunities for further research in this area.

7.2 The Research Question

The rationale for this study was threefold. First, within the literature there is a gap in understanding how teacher beliefs, knowledge and practices interact with one another and previous research designs have not been able to capture the nuances of this phenomenon accurately enough (Muis, 2004; Philipp, 2007). Second, use of multiple representations in school maths can be seen as a highly complex task and there is a gap in the literature in identifying what factors lead to teachers using them effectively (Cramer, Post and delMas, 2002; Rau et al., 2009; Carbonneau, Marley and Selig, 2013). Third, there is currently a national drive from the English government for teachers to develop what is referred to as a 'teaching for mastery' approach and this involves an emphasis on using multiple representations. Therefore, many teachers in England will be attempting to develop their use of representations in school maths lessons (section 2.4.1). I have hypothesised that a teacher's belief and knowledge system is likely to influence the way in which they use multiple representations in the classroom. Therefore, the research question I have asked is:

How do teachers' mathematical beliefs and knowledge influence their use of representations in the process of negotiating the mathematical meaning of fractions?

In answering this question, my study makes four key points: in the case of Gillian, there was a strong connection between her belief and knowledge system and her practice; her 'confident knowledge' of fractions led to a confident use of related representations; the textbook scheme that she used played an important role, especially in relation to mathematical knowledge and representations used; and 'elite' beliefs and knowledge about the basis of success in school maths was closely connected to a 'rhizomatic' use of representations. In this section, I will provide a summary of each of these.

7.2.1 The Connection Between Beliefs, Knowledge, and Practice

Previous research has demonstrated that there is often disparity between teacher beliefs and actual classroom practice (Erikson, 1993; Raymond 1997; Philipp, 2007). In addition to this, other research has highlighted the need to consider how beliefs and knowledge influence classroom practice together, rather than considering them separately (Hill et al., 2008; Sleep and Eskelson, 2012; Kuntze, 2012). Within this study, I developed a theoretical framework that allowed me to do just this (chapter 3). As part of my working hypothesis, I have been able to demonstrate how what on the surface might be considered conflicting or inconsistent beliefs, knowledge and practices, formed a coherent system for Gillian. Specifically, at a global level, Gillian expressed beliefs akin to the fallibilist category as identified by Ernest (1991). She saw mathematics as a subject created by people and one that exists to be used by people. She also maintained such beliefs in a more specific classroom context by emphasising that there were important social traits and learner dispositions that pupils needed to develop to be successful in school maths. Gillian's practice strongly reflected these beliefs, and her teaching was typified by pupils using multiple representations to communicate meaning and learning in a social and autonomous way. Most time in her lessons was spent prompting pupils to share ideas with one another and provide reasons about their thinking. Nevertheless, Gillian also expressed strong beliefs about the importance of pupils learning about specific mathematical knowledge, as identified, and planned for by her as their teacher, and that this would help them with their end of year tests. At surface level, this may seem at odds with the fallibilist stance; because it treats some mathematical knowledge as certain and there to be taken in by the pupils, rather than created by them through social learning activity. However, I argue that this view is akin to what Cobb (2007: 3) describes as researchers who "derive instructional prescriptions directly from background theoretical perspectives" (see section 2.2.2 for more detailed discussion). Within my study, using the LCT specialization dimension enabled me to demonstrate how it is possible for a teacher to hold both ideas together as part of a coherent system of beliefs, knowledge, and practices. This is what I describe as an 'elite' belief and knowledge system. My study has shown how teachers can both allow pupils a high amount of autonomy with regards to how they learn, including the development of their social learning dispositions, but also maintain quite close control of elements of teaching that relate to precise mathematical knowledge acquisition (section 5.5.3). In essence, part of my working hypothesis is that teaching can both be directed and controlled by the teacher, whilst also allowing pupils elements of control over the way in which they learn. Therefore, the first way in which my study has answered the research question is by demonstrating that, in relation to representation use, it is possible for a teacher's belief and knowledge system to closely reflect their actual classroom practice. It is arguable that these beliefs and knowledge are an active part of what led to Gillian using representations in the way that she did. The next section explains how Gillian's specific knowledge of fractions also reflected her use of fraction related representations.

7.2.2 Confident Knowledge and Confident Use of Representations

An important aspect of Gillian's belief and knowledge system was that she had what I describe as a 'confident knowledge' of fractions and their related representations. I argue that this in turn then translated into a confident use of representations within her teaching. This is important as it shows how Gillian's beliefs about the nature of mathematics, and teaching and learning mathematics, were also supported by her subject knowledge. Thus, supporting the conjecture by Sleep and Eskelson (2012) that it is the combination of teacher beliefs and knowledge that influence classroom practice, rather than one or the other. My description of her knowledge of fractions as 'confident' is based upon two things within the data. First, Gillian's knowledge of fractions and the related sub-constructs (Kieren, 1976) was secure. She could identify the different sub-constructs during the teacher problem tasks interview and worked confidently with a wide variety of related representations. Second, she held this knowledge in a confident way. By this, I mean that she was not hesitant in working with the different sub-constructs and their related representations, and when discussing these, expressed firmly that she would happily use them all in her teaching, even if they would be tricky or ambiguous for pupils (section 5.5.5). In this study, I suggest that her classroom practice reflects this confident knowledge as my interpretation of her representation use was also confident in nature. By this, I mean that she was happy to present pupils with representations of fractions that she thought would be deliberately tricky for them. She also used fraction representations as tools for communicating thinking within her lessons and spent significant time doing this in both lessons that were observed. I argue that, in doing this, Gillian displayed a confident use of representations because she was happy to negotiate their meaning with pupils, spending significant amount of time doing this, even if she predicted that pupils would find it hard. Therefore, the second way in which my study has answered the research question is through demonstrating that Gillian having a good knowledge of fraction sub-constructs, and a confident belief in this knowledge, led to a confident and effective use of representations in her classroom practice. Nevertheless, this confident knowledge and confident use was also influenced by the textbook scheme she was using.

7.2.3 The Voice of the Textbook

In answering the research question, through the data analysis it was clear that the textbook scheme that Gillian was using was an influencing factor on her beliefs, knowledge, and practice (section 6.3). In my study I have shown that the textbook was influencing Gillian in two ways: by developing her own belief and knowledge system, and by influencing her representation use. First, I hypothesise that the textbook influenced her belief and knowledge system in that there was a significant congruence between what she espoused as being important and what was in the textbook. Gillian believed in developing social learning and learner dispositions alongside maintaining a precise focus on mathematical knowledge, and the textbook analysis showed that this was also what the textbook authors appeared to be aiming for (section 6.3). The presence of different cartoon characters in the textbook who offered varying ideas and solutions to problems mirrors the way in which Gillian facilitated classroom dialogue in her observed lessons. At the same time, each discrete ‘lesson’ in the textbook had a precise knowledge focus and Gillian’s teaching also mirrored this in that she maintained close control over aspects of her lessons that were about specialised areas of mathematics. Second, I argue that the textbook also influenced Gillian’s representation use (section 6.3). Despite suggesting that she would use a very wide range of fraction representations in an initial interview, in her lessons Gillian only used a select few and these were then ones presented within the textbook. Additionally, the analysis using the LCT semantic dimension demonstrated that the way Gillian used representations throughout her lessons matched the progression that was designed within the textbook. This is important because both things suggest that, where teachers use one textbook or scheme as their main source of mathematical content, it appears likely that this will influence the translation of beliefs and knowledge into actual classroom practice provided that the textbook scheme used has content aligned with these. In essence, my working hypothesis is that, if the content design of the textbook scheme used aligns closely with a teacher’s belief and knowledge system then this seems likely to support the translation of beliefs and knowledge into classroom practice. Therefore, the third way in which my study has answered the research question is by demonstrating that a textbook scheme may influence the way in which belief and knowledge systems influence representation use.

7.2.4 ‘Elite’ Beliefs and Knowledge and a ‘Rhizomatic’ Representation Use

I describe Gillian’s belief and knowledge system as ‘elite’, using the terminology from the LCT specialization dimension. Her belief and knowledge system is typified by seeing the development of precise mathematical knowledge as highly important, whilst also wanting pupils to be able to learn socially and develop certain learner dispositions with autonomy. In this sense, it is ‘elite’ in LCT terms because she both emphasises specialised knowledge and social relations as the basis of success in school maths. Alongside this, I describe the way in which she uses representations in her lessons as ‘rhizomatic’ using the term from the LCT semantic dimension. This means that, for Gillian, successful representation use involved using multiple representations and seeing how they were connected as well as being able to generalise and move away from context specific examples. The design of this study means that it is not possible, nor was it the intention, to say that these two are connected for all teachers. Further research and a different research design would be required to see whether elite beliefs and knowledge about mathematics and mathematics teaching leads to a rhizomatic use of representations in a causal manner. However, this does seem to be the case with Gillian, and it is worth theorising about which aspects of her belief and knowledge system specifically support her representation use. Her belief in the importance of pupils developing specialised mathematical knowledge in a deep and meaningful way seems to be connected to her use of representations. This is because she believes in and uses representations as a way of deepening pupil understanding of the mathematical ideas she is teaching. For Gillian, deep mathematical understanding of fractions is characterised by the ability to use multiple representations and communicate thinking about them effectively. Therefore, the part of her elite belief and knowledge system that emphasises the focussing on precise mathematical knowledge is leading to her using representations in such a way that is congruent with what might be considered effective use within the literature. Specifically, her representation use involves using representations for a precise mathematical point, encouraging verbal reasoning about multiple representations, and encouraging pupils to draw their own representations, all of which are ways of using representations supported by previous studies (Meira, 1995; Carbonneau, Marley and Selig, 2013; Rau and Matthews, 2017). It is also important to highlight that by aiming for pupils to be able to use representations in a rhizomatic way, this also aligns with what Duval (2006) would argue is effective representation use. The ability to use multiple complex, interconnected, and context independent representations is congruent with Duval’s (2006: 111) concept of “treatments” and “conversions” which he argues are crucial for the development of mathematical understanding. Therefore, the fourth way in which my study has answered the research question is by highlighting the type of belief and knowledge system, what I describe as an 'elite' system, that may well lead to an effective, rhizomatic, use of representations.

7.3 Contributions

In answering the research question, I have generated a working hypothesis and been able to make several contributions to knowledge, both theoretical and methodological. In describing some of my contributions as 'theoretical', I am referring to the aim of this study, which was not to make nomothetical generalisations to a wider population (section 3.4), but to generate a working hypothesis about the way in which teachers' beliefs, knowledge, and practices came together to influence the way representations are used to communicate mathematical meaning. Therefore, my contributions to knowledge build upon previous research to contribute to knowledge but can also be used as a springboard for future research as well as potentially informing policy and practice. With regards to my methodological contribution, the application of LCT in the context of a hybrid approach to thematic analysis presents a new and innovative way of studying the beliefs, knowledge, and practices of teachers.

7.3.1 Theoretical Contributions

In this section I will outline my working hypothesis by first discussing the impact of studying beliefs and knowledge together as a 'belief and knowledge system'. Following this, I will discuss the application of LCT by using Specialization to help explain the organising principles of teacher beliefs, knowledge, and practices in the form of an 'elite' belief and knowledge system and by using Semantics to better explain representational activity that is more likely to lead to desirable outcomes as being 'rhizomatic'. Finally, I will explain how the textbook was an entangled part of the way in which mathematical meaning was communicated in the classroom, contributing to the way in which representations were used.

Belief and Knowledge Systems

As discussed previously (section 2.2), within mathematics education research, beliefs and knowledge have usually been studied separately. This has led to there being a gap in the research in relation to how beliefs and knowledge work together to influence classroom practice. My study contributes new knowledge and builds on this gap in previous research by proposing a working hypothesis that teacher beliefs and knowledge should be studied together as a 'belief and knowledge system'. This is partly influenced by Kuntze's (2012) model of beliefs and knowledge and partly influenced by Thompson's idea of belief systems (1992). Building on these ideas, my in-depth analysis of one teacher, grounded in classroom practice and applying LCT, has demonstrated that beliefs and knowledge can be integrated with one another. These seem to come together as a belief and knowledge system, which influences classroom practice. I argue that my study both provides a more useful way of examining teacher beliefs and knowledge, by considering them as a system, and supports Kuntze's (2012) assertion that studying beliefs and knowledge together provides a more accurate picture of how these influence classroom practice. Importantly, my study supports this because I have been able to show that Gillian holds differing beliefs and knowledge at different times, or in different contexts, but that these can come together as part of her belief and knowledge system, rather than being explained as 'inconsistencies' as previous research has done (Philip, 2007). In this way, I argue that beliefs and knowledge are integrated and not separate and come together to influence the way in which teachers act in any particular situation. The most prominent example of this is the way in which Gillian held strong beliefs and knowledge about the importance of pupils developing specialised knowledge in school maths, whilst also strongly believing in the development of social learning and learner dispositions at the same time. Borrowing terminology from LCT, this is what I describe as an 'elite' belief and knowledge system.

An 'Elite' Belief and Knowledge System

Because of the aforementioned gap in the research into mathematics related teacher beliefs or knowledge (section 2.2), the literature review revealed that there is a lack of any theoretical framework that can adequately explain the way in which beliefs and knowledge influence practice. My study makes a contribution to knowledge by addressing this gap and applying LCT to better explain teacher belief and knowledge systems. Using the specialization dimension of LCT, I describe the underpinning principles of Gillian's belief and knowledge system as 'elite'. Maton (2016: 13) describes an elite code as one where:

> …knowing particular specialised knowledge, principles or procedures and having the right sort of personal attributes are both considered to be highly important.

In the case of Gillian, this means that within her belief and knowledge system she holds strong views about the importance of pupils learning specialised mathematical knowledge whilst also having a strong desire for them to develop broader social learning dispositions. Additionally, I describe her as having a confident knowledge of fractions and their related representations. For Gillian, the basis of success in school maths lessons was for pupils to be able to understand the mathematics deeply whilst also being able to act in a certain way, developing resilience, verbal reasoning, social learning, and autonomy. Importantly, she did not emphasise social traits that might be considered 'fixed' and that pupils could not change, such as being naturally gifted, instead the dispositions she was emphasising were things that pupils could actively develop no matter what their starting point. It is important to highlight that, not all the data about Gillian's belief and knowledge system fell into an 'elite' code; there was a balance between this and a 'worldly' code where social relations were emphasised over and above the acquisition of mathematical knowledge. Gillian viewed this as a way of supporting pupils whose affect towards mathematics was not always positive. In this way, she seemed to hold the acquisition of social learning dispositions as a primary belief and the acquisition of mathematical knowledge as a secondary belief. For her, pupils needed certain social dispositions to enable them to gain mathematical knowledge effectively. I argue that the application of LCT answers the call from the literature for researchers to develop better research designs that help explain teacher beliefs so that a more complete understanding can be gained (Philipp, 2007), and therefore offers an important contribution to the development of mathematics education research.

'Rhizomatic' Representational Activity

I also propose, through my working hypothesis, that the LCT dimension of Semantics offers an important contribution to theory about teachers' representational activity in school maths lessons. Specifically, my analysis has demonstrated that a 'rhizomatic' use of representations as the end goal of lessons is likely to lead to effective representational activity in the classroom that supports the development of mathematical understanding. Currently, one of the most common ways of explaining the use of representation in school maths is the so called 'Concrete-Pictorial-Abstract' (CPA) approach (Merttens, 2012), which builds upon the work of Bruner (1966). However, I argue that this approach potentially over-simplifies representation use in school maths, especially given some of the intricate details around what might be considered as effective use of representations for learning (section 2.3). For example, there is contention around what might be considered a 'concrete' representation; is it an actual real-life object, or do physical counters designed purely for school maths purposes also count as 'concrete'? Does the term 'concrete' refer to the physicality of an object, or the real-life contextualisation of it? (See section 2.3.2 for further discussion). As an alternative perspective, my study applied the LCT dimension of semantics to analyse Gillian's representation use. Not only did this enable me to analyse her use of representations in terms of their complexity and contextualisation, but it also assisted the analysis of representation use in a temporal way, across a whole lesson (section 5.7.2). Using this, I explain how Gillian used representations so that the end goal of her lessons was for pupils to understand mathematical meaning in a generalised and de-contextualised way (referred to as a ‘rhizomatic’ use of representations), however, to get there, she tended to begin with context dependent and more simplistic representations. As her teaching progressed throughout her lessons, the complexity of the representations first increased before things became more decontextualised (section 5.7.2). At the time of writing, this is the first time LCT has been applied to school maths research and it offers a more explanatory and in-depth way of studying representational activity in the mathematics classroom. Particularly, I argue that it facilitates a more nuanced approach to understanding how representations are being used by teachers, but also that it offers a new way of analysing the representations provided in schemes or textbooks, which was another important aspect of my study. My working hypothesis is that by moving towards a rhizomatic use of representations as the end goal within lessons, teachers facilitate representational activity in the classroom that is supported by what previous literature suggests is effective use (section 3.3).

The Textbook and the Teacher Collaborating

My analysis demonstrated that the textbook seemed to play an important role in influencing Gillian’s beliefs, knowledge, and practices. This forms another theoretical contribution to knowledge within my working hypothesis; the case of Gillian demonstrates how it is possible for a school maths textbook to support the recontextualisation of mathematics into school maths whilst straddling the official recontextualising field (ORF) and the pedagogical recontextualising field (PRF) (see section 2.4 for a full discussion about these). The textbook analysis made it clear that the textbook authors were interpreting the official National Curriculum for England (DfE, 2013a); they were “selecting an offer of meaning” in the way that they had broken down the mathematical content into lessons and selected specific representations (Lilliedahl, 2015: 41). In this way, the textbook that Gillian was using was an official document and therefore part of the ORF (Bernstein, 2000). Nevertheless, my study also demonstrates that the textbook influenced the way in which Gillian selected and used representations, to the extent that it appeared to temper the variety of fraction representations she used in her actual lessons, compared to what she espoused during the teacher problem tasks interview. It was influencing the way in which she recontextualised mathematics through her pedagogy and was also therefore part of the PRF (Bernstein, 2000). In essence, I argue that the voice of the textbook joined together with Gillian's own beliefs and knowledge to influence the representational activity within her classroom. Using Barad's (2007) theory of agential realism, the textbook, as part of the physical 'matter' of the classroom, was an integral part of the meaning being communicated in Gillian's classroom. My working hypothesis is that the textbook came together with Gillian's beliefs and knowledge system in an entangled way to create the meaning that was being communicated in her lessons.

7.3.2 Methodological Contributions

My adoption of a three-phase "hybrid approach" to thematic analysis (Fereday and Muir-Cochrane, 2006: 82), particularly utilising the idea of retroduction from critical realism (Crinson, 2007) and applying LCT as a part of this, offers a methodological contribution to mathematics education research. Previous research about teacher beliefs and practices, where the focus was on in-depth data collection around one or two teachers similar to my case study (Erikson, 1993; Raymond 1997), did not apply such a rigorous approach to data analysis and therefore suffered from methodological weakness, leading to findings that were not able to fully explain the data generated. For example, Raymond (1997) applied a theoretical framework to the data to try and understand the connection between the beliefs and practices of a single teacher. However, her study concludes that teachers have inconsistencies between their beliefs and practices rather than being able to explain the connection more fully. Within my study, I argue that the three phased approach to data analysis (section 4.8), using a mixture of inductive and deductive reasoning, has enabled me to come closer to closing the so called "discursive gap" between theory and empirical data (Bernstein, 2000: 445). By first taking the time to get to know the data itself, without applying any external theoretical framework, and then applying the data instruments and then LCT, I was able to analyse the data corpus from multiple perspectives and offer a more detailed explanation of beliefs, knowledge, and practices than previous research has done. My study supports the adoption of this approach to analysis for future mathematics education research, some suggestions of which will be made later in this chapter (section 7.6).

7.4 Research Design Reflections and Recommendations for Further Study

In this section, I will outline some of the strengths and confines around my research design, offering suggestions for future research that builds on my study. First, the choice to focus on a case study of one teacher will be discussed with reference to my stance towards the generalisability of findings. Second, I will offer some reflection on my positionality as an insider researcher and how I maintained a critical distance between my professional role and my role as a researcher. Third, the absence of any 'pupil voice' in my study and what the implications for future research are in this area will be outlined. Finally, reflections on the context in which my case study took place alongside suggestions for further research specifically relating to the use of textbooks will be discussed.

7.4.1 Generalisability of Findings

First, I argue that one of the key design strengths of my study was the decision to focus on a case study of one teacher. Doing this enabled me to analyse a broad range of data in an innovative way, adopting my three-phase hybrid approach to analysis of data, that was grounded in classroom practice. When designing my study, there was a possibility of doing a multiple case study with a larger sample size instead (approximately four teachers – see section 4.3 for further discussion). Although doing this may well have generated more variation within the data corpus, I would not have been able to analyse the amount of data that I collected for just one teacher in such a detailed way, had I focussed on more teachers. By focussing on one teacher in-depth, I was also able to create an approach to analysis that captured multiple perspectives of teacher beliefs, knowledge and practices, accounting for greater ontological depth, in line with critical realist methodology (Olsen, 2009). Ultimately, the decision to focus on just one teacher relates to my stance on the purpose of my study, specifically the approach to generalisation, which has been discussed previously (section 3.4) but merits a final comment here. Often, education research is treated as either producing findings seen as rules or laws that are generalisable to a wider population (so called 'nomothetic' generalisation), or as being completely limited to the direct context of the research data (so called 'ideographic' generalisations) (Cronbach, 1975; Lincoln and Guba, 2009). Both stances are problematic. Nomothetic generalisations assume that knowledge is absolute and decontextualised and therefore ignore the influence of culture and society. Ideographic generalisations dissolve knowledge into simply a process of knowing, leading to what Maton (2014) refers to as knowledge blindness (see chapter 3 for detailed discussion). Within this study, I aim to avoid the pitfalls of either approach and generate a "working hypothesis" that can contribute to theory development (Lincoln and Guba, 2009: 38; Yin, 2014). Essential to this has been the adoption of LCT as part of my theoretical framework (chapter 3), which has enabled me to identify the practices and dispositions of a single teacher and place these in terms of LCT dimensions offering an explanation of the organising principles that lie behind her beliefs, knowledge and practices. As mentioned previously, it was the precise focus on one individual teacher that enabled me to go into such depth with the data analysis and offer some contributions to knowledge in the form of a working hypothesis (section 7.3).

7.4.2 Researcher Positionality

As is outlined in section 4.2, I consider myself to be an insider researcher, meaning that I have a professional role within the context that I am studying. This had important implications with regards to my reflexive stance and how I managed to balance my dual role of both educational professional and researcher. Currently, within England there are conflicting processes underway in the realm of school maths development whereby system leadership is promoted by the government (designed to promote school and teacher autonomy) whilst state controlled professional learning programmes are rolled-out with increasing central control (see section 4.2). This is what some refer to as "state-market assemblage" (Boylan and Adams, 2023: 3). In my professional role as a Maths Hub lead, I may be considered as a policy mediator involved in the translation of state controlled professional development programmes into enacted experiences for teachers. Although the implementation process through which this is done arguably offers a sense of balance for professional autonomy (through lesson study and critical reflection of the 'teaching for mastery' policy), it was important that I maintained a self-reflexive stance throughout my study to ensure differentiation between my professional role and my role as a researcher. In doing this, I adopted two key methods that are worth reflecting on, member checking and the use of a reflective journal.

I found the process of member checking to be a complex one that required careful negotiating, in line with suggestions made by Hallet (2012). Nevertheless, it was an integral part of my research design and helped me enhance the voice of Gillian throughout the analysis of the data. This process was particularly useful in helping me better understand how my own beliefs and knowledge of school maths within England were influencing my interpretation of the data. One recommendation for future study is that there is potential for a greater degree of member checking. For example, it would be possible to engage teacher participants in a joint textbook analysis alongside the researcher.

Alongside member checking, I also found that the use of a reflective journal throughout my study was a useful way to maintain high levels of reflexivity. The approach I took was to use writing as a method of inquiry about myself (Richardson, 2000), using an approach like that described by Watt (2007). This was a habit that I had used prior to this study during previous research and was therefore something that I could naturally pick up and use, rather than it being a brand-new experience for me. This technique was especially useful during the analysis and writing up phases of my study. I was able to look back on my journal entries and consider how my on-going reflections influenced my thinking. In many ways, this approach was more than just a useful tool in conducting my study, and it became an important way in which I could be more aware of my own professional development throughout the research process. I now see the use of a reflective journal not just as a tool to deploy during a research study but as an ongoing way in which I can develop in my identity as an educational researcher alongside my professional role within the English education system.

7.4.3 Pupil Voice

The second reflection here is the absence of any ‘pupil voice’ from my case study. Although both direct observation and video recordings of pupils in lessons were central parts of the data corpus, the emphasis was on the teacher. This is because the main purpose of this study was to better understand the role of the teacher; how teachers use representations to communicate mathematical meaning, and how their beliefs and knowledge are entangled with this. This is not to say that the pupils are ‘missing’ from my study, simply that I acknowledge the central role that the teacher plays in creating lived experiences for pupils in school maths lessons. Previous research has shown that the way in which teachers use representations can have a direct impact on how pupils learn (Cramer, Post and delMas, 2002; Carbonneau, Marley and Selig, 2013; Tunç-Pekkan, 2015; Rau et al., 2017; Rau and Matthews, 2017), and also that teacher beliefs and knowledge are likely to influence the way in which they teach which, in turn, will influence the way pupils perceive mathematics (Erikson, 1993; Kloosterman and Cougan, 1994; De Corte, Op’t Eynde and Verschaffel, 2004; Muis, 2004). Therefore, in my study, I take the stance that studying what the teacher thinks and does is of central importance to beginning to understand how pupils are impacted. Nevertheless, future research gathering more nuanced data from pupils, such as interviews, focus groups, and focussed video recordings of them in maths lessons, would enable a more fine-grained analysis of how pupils themselves perceive school maths in relation to the LCT dimensions. For example, an interesting study would be to analyse pupils' representational activity and their perceptions of representations in a maths lesson in more detail and then compare this to the textbook scheme being used, and then the teacher's representational activity. Using the LCT dimension of semantics to do this would reveal the extent to which the intentions of the teacher and textbook scheme translated into actual pupil perceptions. It would be interesting to then break this down into different pupil sub-groups. For example, focussing upon pupils with lower prior-attainment, or those who the teacher perceives as struggling, asking the question, 'Is the way in which they use and perceive representations different from others throughout the lesson?' The LCT specialization dimension could then also be used to analyse their perceptions of what is required of them to be successful in school maths lessons, asking the question 'Do their perceptions of the basis of success align with the teacher's intentions?' This would build on the findings of my study and contribute to more deeply understanding the phenomenon of representational activity in the primary mathematics classroom.

7.4.4 Research Context and the Textbook

The third and final reflection here relates to the role of the textbook in the school maths classroom. As has been outlined previously (section 1.2), my case study has been conducted within a school where there has been significant professional development funded by the UK government on what is known as 'teaching for mastery'. Two significant aspects of this are the widespread adoption of government approved textbooks and the development of a wider use of representations. This means that the findings of my study must be interpreted with knowledge of this context, and application of these findings to teachers where there has not been such significant professional development in these areas would be unwise. Despite this, one important finding was that Gillian used the textbook as her main source of mathematical content, with barely any additional content coming from other sources. This is at odds with what has previously been found in England, where it was reported that teachers only use a textbook as the basis of their instruction 10% of the time (Mullis et al., 2012). Nevertheless, given the more recent government approval of primary maths textbooks, new research is underway to try and establish whether this shift in textbook use that I found within my study is more widespread within England (Barclay, Barnes and Marks, 2022). Given this, alongside the important role that I have identified the textbook in playing with regards to the connection between teacher beliefs, knowledge, and practice, it also merits further study in a more detailed way. Future research could further develop Hetherington and Wegerif's (2018: 27) "material -dialogic pedagogy", considering specifically primary mathematics teaching using a textbook. Using fine-grained video analysis of lessons where a textbook is used and applying this theory would allow for a more detailed understanding of how the textbook plays a part in the communication of mathematical meaning than I have been able to do in my study.

7.5 Implications and Recommendations for Policy and Practice

One of the aims of my study has been to influence educational policy in England and the classroom practice of teachers, therefore it is important to outline how the findings may be used to make this happen. There are three key ideas that this study can contribute towards this: considering a knowledge rich curriculum and whether this automatically implies a pedagogical approach; the connection between beliefs, knowledge, and practice; and using the semantic dimension of LCT to better understand representational activity in the classroom. The final two sections here refer to on-going work that I have been doing with primary school mathematics teachers in England for the past year, using some of the findings of this study to better help them engage with these issues.

7.5.1 A ’Knowledge Rich’ Curriculum and Associated Pedagogical Approaches

This first recommendation is relevant to those in the position of forming policy at a national level and who are central to the process of recontextualising mathematics into school maths as part of the official recontextualising field (Bernstein, 2000). Part of the context of this study has been a national drive to develop a ‘knowledge rich’ curriculum within English primary schools (DfE, 2013a; Gibb, 2021). There has been a growing policy within English education that curriculum subjects must have an unwavering focus on the development of specialist knowledge, and this has been associated with so called ‘teacher-led instruction’ (Gibb, 2017). A ‘teacher-led’ approach is contrasted with a ‘child-centred’ approach where teachers try and develop learner dispositions such as creativity and problem solving “as if these skills transcend domains of knowledge” (ibid., 2017). Arguably, the un-written assumption is that for there to be a focus on knowledge acquisition, then certain pedagogical methods are more appropriate than others. Nevertheless, my working hypothesis suggests that, with regards to school maths, this is not quite so straightforward. In the case of Gillian there were parts of her beliefs, knowledge and practices that were very teacher control centred and these were related to the development of specialised mathematical knowledge. However, she also strongly believed in the development of social learning and learner dispositions and her classroom practice reflected this. Therefore, I argue that this dichotomising of ‘teacher-led’ and ‘child-centred’ learning against one another is unnecessary. The growing national policy shift where the two approaches are pitted against one another is unhelpful for teachers and could either lead to a backlash where the importance of knowledge is rejected and teaching succumbs to “knowledge blindness” (Maton, 2014: 4), or teachers begin to assume that the development of social learning and learner dispositions are not important. Neither of these outcomes seem desirable given that it is possible, as my working hypothesis suggests, for teachers to avoid the dichotomy altogether and develop both in tandem, opting for an ‘elite’ belief and knowledge system about what it means to be successful in school maths. There is a need to make it clearer within national educational policy that a knowledge rich curriculum does not preclude the development of social learning and learner dispositions and, at least, in the case of primary mathematics it may well be desirable that both are emphasised within teaching.

7.5.2 Connecting Beliefs, Knowledge, and Practice for Teachers

This second recommendation relates both to practising teachers as well as those in the role of leading professional development for teachers. As has been outlined previously, one of the contributions of my working hypothesis is that studying beliefs and knowledge together as a system is a more effective way of understanding how these might influence practice (section 7.3.1). Although this is a complex area, this study shows that it is of importance in terms of what influences teacher practices and is therefore an important issue for teachers to reflect upon. To get this process of reflection started, teachers may consider the following two questions: ‘What is mathematics to you?’ and ‘How do you think parents of pupils would answer that question?’ In my experience, these two questions often prompt significant discussion about mathematics being ‘used by everyone’ and something that is ‘everywhere around us’. Additionally, the common perception amongst teachers is that typical parents of pupils in English schools do not see mathematics like this and instead see it as a set of facts and procedures to be learned. In drawing out these discussions with teachers, it is then possible to contrast these two views about the nature of mathematics and start to relate them to the idea of ‘fallibilist’ and ‘absolutist’ philosophical stances as outlined earlier (section 2.1) (Ernest, 1991). This could then lead into a discussion about what classroom practices and resources (such as textbooks) might enable desirable beliefs and knowledge to be put into practice.

In my experience, this type of activity conducted with teachers facilitates a greater awareness of what their beliefs are about the nature of mathematics and learning mathematics, what their own professional knowledge is like (and where they might lack some knowledge of certain areas of mathematics), alongside whether their own classroom practice reflects the things they believe and know about the subject. Following on from these discussions, it is then possible for teachers to analyse practice, whether through video recorded lessons or real-life observation, and consider how what is going on in the classroom reflects what they believe about the nature of mathematics and how pupils should experience it. I argue that prompting these discussions not only makes philosophical arguments about the nature of mathematics more accessible to teachers but is also likely to lead to school maths that does not suffer from “low epistemic quality” (Hudson, Henderson and Hudson, 2015: 377), because teachers are more aware of their own beliefs and how their knowledge and other resources might influence the translation of these into classroom practice. Not only this, but by engaging teachers in this discussion, it also encourages them to better understand their role in the process of recontextualizing the subject of mathematics into school maths as part of the pedagogic recontextualising field (Bernstein, 2000).

7.5.3 Using LCT to Help Teachers Analyse Representations

This third recommendation also relates to practising teachers and those providing professional development for teachers. As mentioned previously, one of the most common ways in which primary school teachers in England think about mathematical representations is through the ‘Concrete-Pictorial-Abstract’ (CPA) approach (Merttens, 2012). Previously in this study I have argued that, although this approach is a useful gateway into thinking about representations, it potentially over-simplifies some important issues (see section 2.3.2 for further discussion). Therefore, another way in which my study can influence the practice of teachers is by using the LCT semantic dimension to analyse representational activity in a more nuanced way. This is something I have already begun to trial when working with teachers, who are specifically using a textbook.

Initially it is useful to engage teachers in a conversation about the nature of mathematical objects, asking them to think of as many representations as they can for a number and then get them to consider whether any of them are the actual number itself. This initial activity generates discussion about the nature of mathematical objects and how it differs from most other primary school subjects because it is entirely abstract in nature and even the most basic ideas that we teach pupils can be represented in multiple ways. Once the abstract nature of mathematical objects has been established, the LCT semantic dimension can then be used to analyse an example of a textbook representation. As an example, see figure 23 below.

Figure 23 - Image of a textbook fractions problem, taken from the Maths No Problem! series.

This might prompt a discussion about the fact that it is sort of 'concrete' in that it is a real-life object (chocolate) but also not, because it is a cartoon-like representation. It is important to consider whether the image is even mathematical at all. We know that it is designed to be, because of its existence in a mathematics textbook, but presented outside of that context it could be considered just to be a cartoon of people sharing some chocolate; it requires the person looking at it to make it mathematical. A simplified version of the LCT semantic dimension plane can be then used to consider a different way of analysing this representation (figure 24).

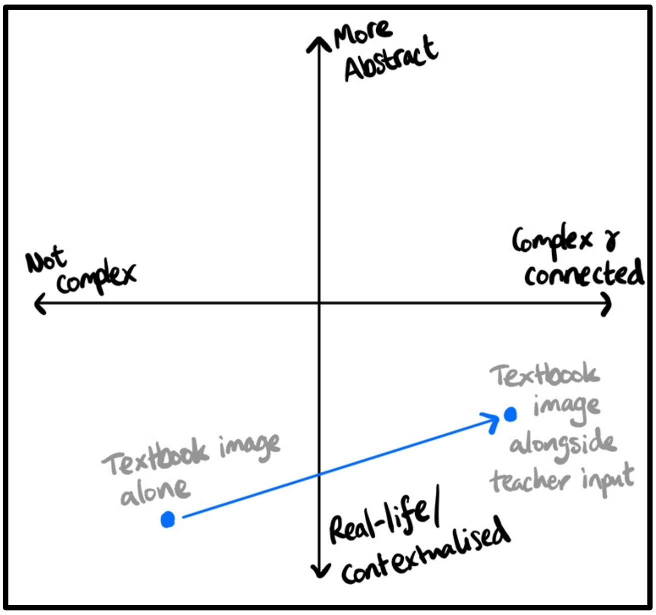


Figure 24 - A simplified version of the LCT semantic plane, with the type of annotation made by teachers after analysing a textbook image

By asking teachers to place the textbook representation on the semantic plane, it prompts discussions about how the representation is used. For example, it is likely that fraction symbols would be introduced early in the lesson as this is the eighth lesson in a fractions chapter. Therefore, the complexity of meaning would shift considerably as the lesson progresses. By doing this, arguably teachers are reflecting not only on the nature of mathematical representations themselves, but also on the importance of perception. What I mean by this is that the complexity and interpretation of any representation is dependent on the way it is perceived by a person, thus engaging teachers in the complex issues around representation use, in particular the social constructivist arguments about how representations are used to communicate mathematical meaning (see section 2.3.3). Alongside this, I argue that this is also another important aspect of getting teachers to engage in thinking about their role in the recontextualisation of mathematics into school maths as part of the pedagogic recontextualisation field (Bernstein, 2000), mainly because the way in which representations are used to communicate meaning are an important part of how pupils experience the subject and is a part of making sure that the subject does not suffer from “low epistemic quality” (Hudson, Henderson and Hudson, 2015: 377).

7.6 Final Concluding Thoughts

In sum, my study sought to better understand the phenomenon that is teachers' use of mathematical representations and how their belief and knowledge system influences this. In doing this, the aim was to be able to contribute to theory through the creation of a working hypothesis, but also to influence policy and practice. I wanted the findings of my study to be able to directly support teachers in their pursuit to help pupils learn mathematics more effectively, as well as driving mathematics educational research forward. Because of my position as an "insider" researcher (Hellawell, 2006: 484), I have already been able to use the findings to create tangible outputs in the form of teacher professional development. In doing this, I have become increasingly convinced that the design of my study has enabled me to avoid the so called "discursive gap" between theory and data (Bernstein, 2000: 445), because I am now able to use the findings to help teachers reflect on their actual practice of using representations. However, this is connected back to the theoretical framework of LCT, in particular the semantic dimension, as is described in section 7.5.3. Thus, I can use my findings to help teachers see how their daily practice of using representations is connected to an underpinning theory, which can start to explain the socially created "rules of the game" in the primary mathematics classroom (Maton, 2016: 3). I have also been able to demonstrate how the resources that teachers use (in this case, a mathematics textbook scheme) potentially play an important role in influencing both belief and knowledge systems as well as actual practice. This is of particular importance within England given the recent government recommendation and funding for schools to use specific textbooks.

Although in my study I present some important findings, in relation to applying the LCT dimensions as theoretical tools, it is just the beginning. As is outlined in section 7.6, my study has somewhat opened the gate for a broad range of future research to take place that will contribute to better understanding what goes on in the school maths classroom. It is my intention to continue to utilise the LCT framework as a tool to analyse and understand the school maths classroom and use this to continue to support teachers in developing their practice.

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9 Appendices

9.1 Appendix 1: Interview Schedule

Interview 1

This interview involved Gillian going through and completing the ‘teacher problem tasks’ (section 4.7) and discussing each one.

Interview 2

Contextual Information

Name: Date:

Number of years teaching:

Mathematics related qualifications / professional accreditations:

1. **What is mathematics?** Please read the two different answers below and provide some comment on them and what your own answer to the question would be.

It’s all about right and wrong isn’t it? It’s different from something like English because there are definite correct ways of doing things. You have to learn different procedures and strategies to get the correct answers to questions.

> It's a way of explaining the world around us. Maths can be used to help us understand things and we have to be quite creative with it to solve different problems in the world. Really, it's all about patterns and using these to explain things.

2. (Follow on from the previous question) **What do you think the implications for teaching maths in school is then?**

3. (Focus on lesson video) **Please watch the video and pause it to make comment about the way in which you are using representations.**

4. (After the video) **How representative is this of the way in which you would normally use representations?**

Interview 3

1. **How did you plan for the lesson today? What about the use of representations specifically?**

2. Focus on lesson video) **Please watch the video and pause it to make comment about the way in which you are using representations.**

3. (After the video) **How representative is this of the way in which you would normally use representations?**

Follow up prompts (to be used to clarify or gather further information about an answer):

- Tell me more about…
- What kind of… is…?
- I've noticed that… please can you tell me more?
- Is there anything else about… ?

Interview 4 (added after initial data collection and early analysis had taken place)

This interview involved a process of member checking, where I asked Gillian to comment on my initial analysis and then I also asked the follow-up questions below.

1. **What do you think it takes for a pupils to be good at school maths?**

2. (based on response to previous question) **How do you deal with it when a pupil is not displaying these characteristics?**

9.2 Appendix 2: Lesson Observation Schedule

Date/time: **Teacher:**

Year group:	**No. of pupils:**	**Maths focus:**	**Other pertinent contextual data:**

Time (mins)	**Teacher Activity**	**Pupil Activity**	**Notes on Representations Used**	**Other Comments**
0-10				
10-20				
20-30				
30-40				
40-50				
50-60				

Themes

These are the specific aspects of effective use of representations taken from the literature as outlined in the 'Data Instruments' section of the Theoretical Framework. They should be used as a guide as to what to look out for and take note on during the observation, but the observer should still try and make a full account of the lesson.

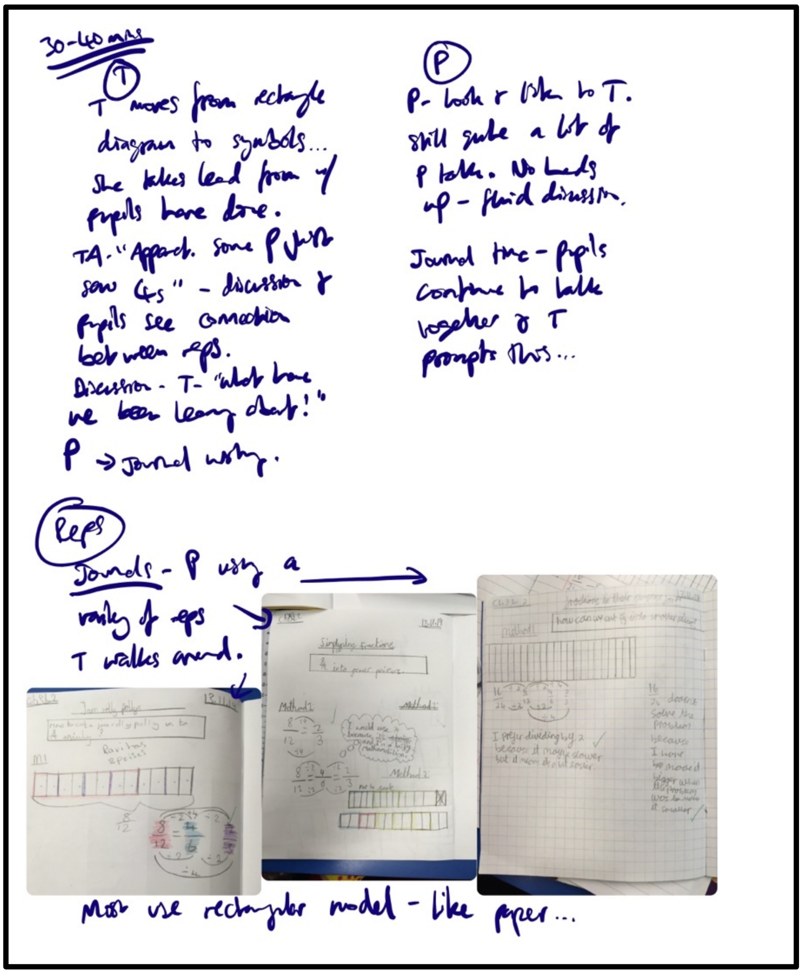
a) Teachers use more than one representation and help pupils to make connections between these (this includes symbols, spoken and written language and physical gestures used to represent a mathematical idea).
b) Representations are treated as discussion points in their own right (rather than just a means to an end) and reasons for using them are made explicit to the pupils.

c) Teachers are explicit when making translations between one representation to another.
d) Teachers allow time for pupils to develop their own explanations about representations that are being used.
e) Teachers allow opportunities for pupils to develop their own representations.
f) Representations are used with a clear mathematical purpose.

9.3 Appendix 3: Exemplar Lesson Observation Field Notes

The following example demonstrates the way in which the above lesson observation framework (Appendix 2) was used to collect data. Two examples are show below. One is a screenshot of the original field notes, which were hand-written using an iPad and photos were inserted in real-time during the lesson. The other is the typed-up version of the field notes. These were written on the same day as the lesson observation and the lesson recording was used to develop the amount of detail that was included. I also began to write reflections about what I had seen in the lesson within the typed-up version.

Example hand-written field notes (excerpt taken from lesson observation 1):



Example typed field notes (excerpt taken from lesson observation 1 – same section as the above example):

30-40	After looking at the diagrammatic solution, T moves on to looking at the symbolic/abstract solution that some pupils have used (dividing numerator and denominator). At this point, the teaching assistant calls out "Apparently some pupils just saw the 4s…" – this seemed to prompt pupils to make an explicit connection between the symbolic solution to the problem and the diagrammatic one. Following some discussion about this, T asks "So, what have we been learning about" – prompting some discussion about the lesson objective. T then asks them to write on their journals about what they did.	Pupils are looking and listening to the teacher however there is still a fair amount of quiet talk going on amongst pupils. This seems to be accepted by the teacher. It also seems to be the case that there is a fluid approach to whole class discussion – pupils do not put hands up and the teacher accepts pupils who call out with ideas or comments. Pupils respond by calling out a range of things like "simplifying fractions", "simple fractions" or "jam rolly polys" Pupils quickly respond and begin to write in journals – see images.	In their journals, pupils seem happy to use a variety of representations and whilst they are doing this the teacher is walking around having individual discussions with them about what they are doing.	There are high levels of pupil collaboration and discussion the whole way through this section despite it being mostly whole class discussion and not peer to peer work.

9.4 Appendix 4: Textbook Analysis Framework

Horizontal Analysis

Name of Textbook	
Number of Chapters	
Number of Pages	
Size of Pages	
Progression of Chapters (including number of 'lessons' in each chapter)	
Other Pertinent Characteristics of Textbook	

Vertical Analysis - Fractions

The unit of analysis is each mathematical problem (including worked examples) in the textbook. So, an 'In Focus' task combined with its ensuing worked example counts as one task. Any other numbered task or worked example also counts as one problem.

Representations	*Textbook*	*Workbook*
Area Model		
Area model combined with a quantity		
Quantity image alone		
Linear		
Decorative Only		
Linked to Context but not Mathematics		
Linked to Context and Mathematics		
Prompted use of manipulatives		
None		
Construct		
Part-part whole		
Operator (Transforms lines, figures or numbers)		
Ratio (comparison)		
Measure		
Quotient (Division of two whole numbers)		
Part-part whole and Operator combined		

Progression Commentary:

9.5 Appendix 5: An example of the thematic coding process showing how one theme was generated

To exemplify the thematic analytical process, one theme has been chosen and the process through which this theme was generated is described below. The chosen theme is 'Learning school maths requires resilience and reasoning'. This theme was chosen as it was one of the trickiest to generate and therefore exemplifies the fine-grained approach to this stage of data analysis. This description will begin by explaining the initial grouping of codes, including brief description of these codes and then provide rationale for the theme itself.

Code Name	Code Description	Number of Quotations
Teacher Beliefs About Mindset in Maths	This code is about Gillian's beliefs about what sort of mindset she wants pupils to develop. Specifically, she asserts that pupils will do better at school maths if they have the appropriate mindset. To her, this means that pupils are resilient, understand how maths relates to their lives, are curious and independent learners and see maths as requiring communication between one another (proving ideas) rather than just getting answers. This code is closely linked to the code 'Teacher Beliefs About the Nature of Maths'.	11
What Makes a Pupil a 'Struggler' or 'Advanced'	This code is all about the different way Gillian seems to differentiate between learners - this seems to be quite broad because at times she refers to them by 'ability', or 'strugglers' and 'more advanced pupils' however she also seems to differentiate between them in terms of resilience and confidence to prove their ideas using a combination of representations and reasoning. She has quite a broad, holistic view of them as learners. This speaks to her view of what it means to be good at school maths and how she assesses pupils as advanced or struggling.	15
Maths as a Creative Subject	This code was difficult to analyse. Although it was only apparent in a small part of the dataset, it seemed important enough to merit it's own code. It is all about Gillian's belief in school maths being a subject in which pupils can be creative. She relates this mainly to a creativity in the process of doing maths (e.g., using representations, proving, and justifying ideas) rather than any sort of creativity with answers to mathematical questions.	2

Using Representations and Pupil Mindset	This code is closely related to teacher beliefs about mindset and is somewhat a 'sub-code' for it. This code is all about how Gillian specifically uses representations to support her beliefs about mindset. To her, representations are not just used for learning mathematical concepts, they are also used to support the affective environment for pupils. She uses representations to help pupils get in the right frame of mind to learn. This is also connected to the code 'Purposeful Use of Representations'.	4

Identification of connected codes

Initially, four codes were identified that all seemed to relate to the different ways in which Gillian thought about the pupils in her class in relation to her maths lessons. The four codes are outlined below with brief description alongside the number of related quotations from the original data.

Rationale for Generating the Theme

The four codes outlined above were used to generate the theme 'Learning school maths requires resilience and reasoning'. The main rationale for this was that each code presented something that specifically spoke to the nature of what Gillian believed to be important for pupils as learners of school maths. Nevertheless, when I was originally coding the data, I could see that these were all connected because of their relevance to beliefs about pupils in maths lessons, however I had thought that they would each become part of different themes, rather than constituting a theme themselves. Despite this, through the thematic analysis process I realized that I was trying to force the data to fit my own template, rather than accepting something that my own coding was showing me. Therefore, I realized that these needed to become a code on their own.

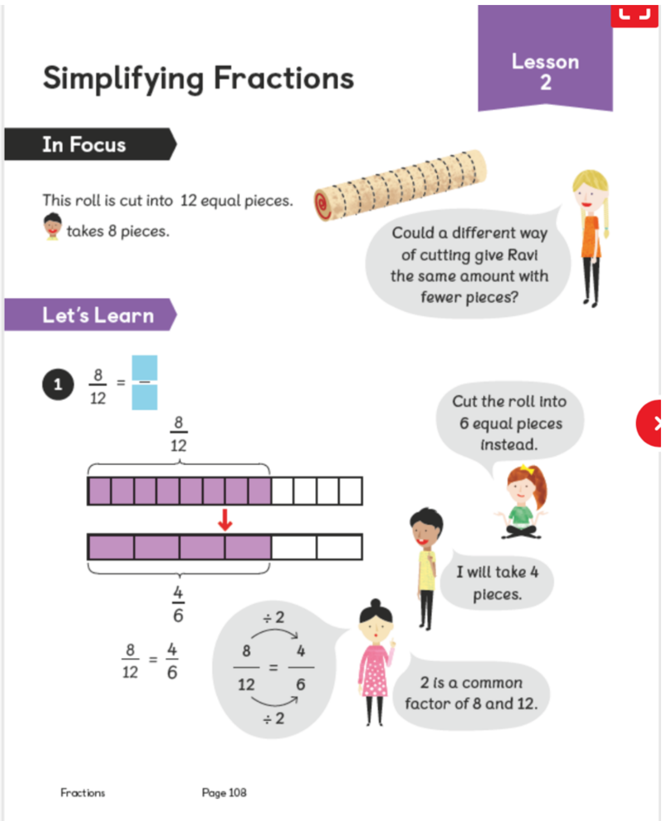
Another issue that became apparent was that the codes within this theme were closely related to many of the other themes. For example, the code 'Using

Representations and Pupil Mindset' contributes an important aspect to this theme, but also relates closely to the theme 'Using Representations for Mathematical Thinking' and therefore contributes to both. Initially, I tried to subsume the codes within this theme, into one or other themes. However, each time I tried to do this, I felt that some of the important information that it highlighted about Gillian's overall beliefs was lost. By keeping it as a theme, I felt that it provided a useful and important aspect to the analysis that ultimately helped answer the research question.

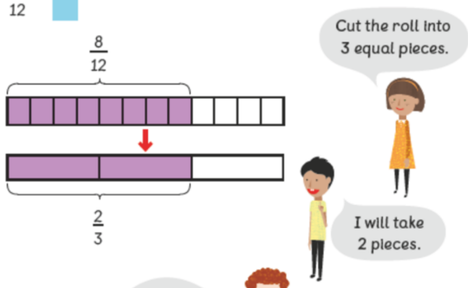
9.6 Appendix 6: A detailed account of each observed lesson

9.6.1 Lesson Observation 1

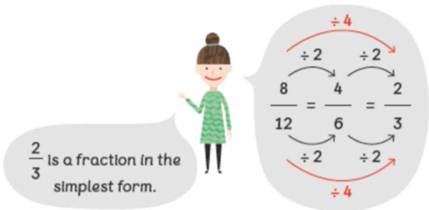
This first lesson occurred in early November 2019 and was the second lesson in a unit on fractions that the teacher had just begun teaching. The focus was on simplifying fractions into their simplest form using common denominators. The first lesson, which they had the day before, was also focused upon the same thing. The images of the Maths – No Problem!™ (Ban Har et al., 2014) textbook pages below are what Gillian had used to inspire her planning for the lesson. She also used a significant number of these images in the lesson itself, displayed upon the interactive whiteboard. Having seen the textbook pages beforehand, I could understand what the lesson was about and generally where it was headed in terms of content right from the beginning.



2 $\frac{8}{12} = \frac{\square}{\square}$



$\frac{8}{12} = \frac{2}{3}$



Guided Practice

1 Write each fraction in its simplest form.

(a)

$\frac{9}{12} = \frac{\square}{\square}$

(b)

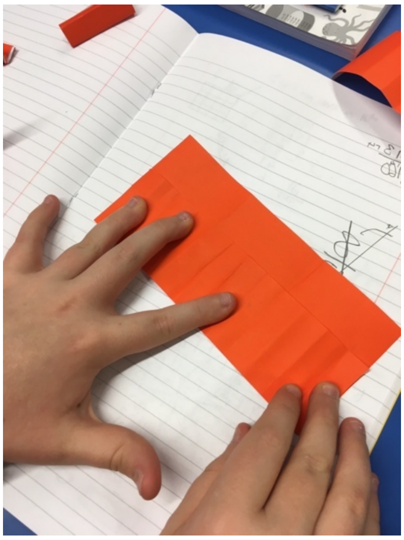
$\frac{6}{10} = \frac{\square}{\square}$

2 Which fractions are in the simplest form?

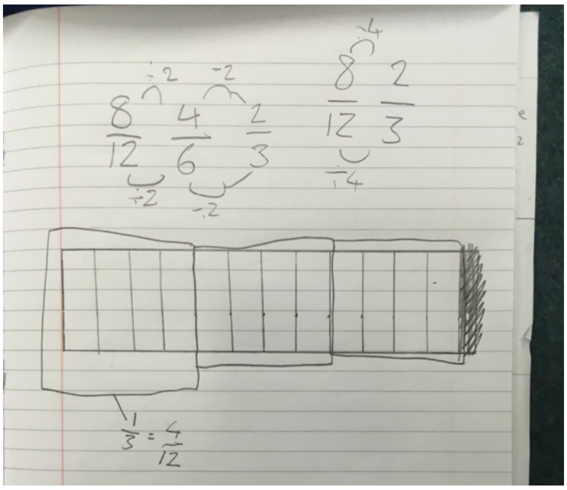
$\frac{6}{7}$	$\frac{6}{8}$	$\frac{6}{12}$
$\frac{8}{10}$	$\frac{8}{11}$	$\frac{8}{12}$
$\frac{5}{12}$	$\frac{7}{12}$	$\frac{10}{12}$

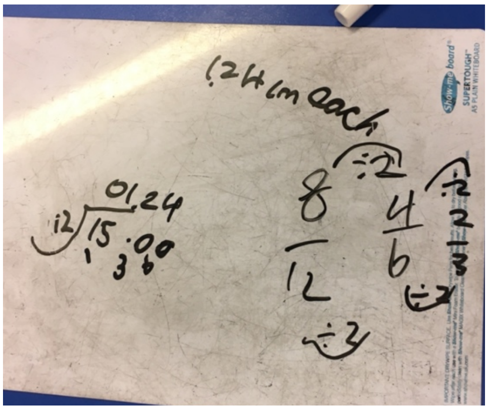
Complete Worksheet **2** – Page **61 - 62**

Gillian started the lesson by showing the picture from the first page of the textbook (the ‘In Focus’ task) with the text covered up so that the only image the pupils could see was the jam roll and the two characters. She provided each child with their own strip of paper and asked them to imagine that it represented the jam roll and to fold their paper in the same way the jam roll has been split up, into 12 equal parts. Pupils took quite a lot of time doing this and were discussing what they were doing amongst one another. Some pupils struggled to fold the paper and required support from the teacher. The image below shows an example of a pupil’s folded piece of paper. Observing this part of the lesson, I questioned whether this was a mathematically useful activity. Later, during the following interview, Gillian explained that she was doing this more to get the pupils comfortable with the idea of learning about fractions rather than engaging them in mathematical thinking.

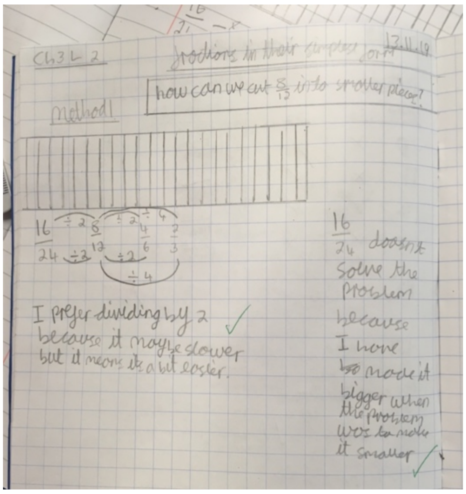


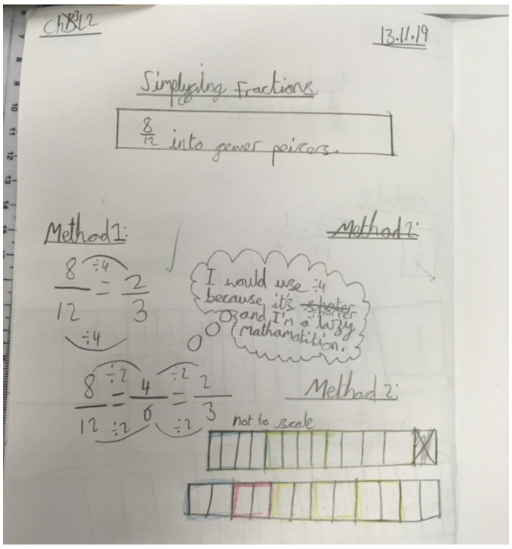
After about ten minutes, Gillian then revealed further text that related to the jam roll image, which referred to a character taking 8 pieces. Almost immediately a pupil shouted out that this is “eight twelfths” and Gillian wrote this on the board as ‘$\frac{8}{12}$’. There was also a written question on the board asking whether he (the textbook character) could have the same amount of roll with fewer pieces (see textbook image above). Gillian asked the pupils to try and figure this out and they spent ten minutes doing this alongside drawing their own diagrams in notebooks and discussing their ideas about how to solve the problem. As an observer, it interested me that such a large proportion of the lesson was spent studying just one problem from the textbook pages. At this point in the lesson pupils were using jotters and mini whiteboards to make notes (see images below as examples) and there was a lot of talk between pupils. I could see that the pupils were not only enjoying the lesson but were actively engaged in thinking about the content. During this time, Gillian was walking around the classroom watching what pupils were doing and occasionally interjecting to have discussions with them.





After the pupils had time to explore this collaboratively, Gillian called them all together to start discussing what they had done by asking, “Does anyone want to explain what they have done to find out?”. During this part of the lesson, Gillian was listening carefully to what pupils were saying and then reflecting what they said back to the whole class, whilst creating her own version of representations on the whiteboard for all pupils to see. At this point, I noted that there were a combination of rectangular area model and abstract symbolic representations being used. As an observer, I could see that multiple representations were being used but also that the whole lesson so far had been coherent, and all the representations used worked well together. The conversation at this point started off with looking at how pupils had used a diagrammatic method to solve the problem before moving onto a symbolic representation of the solution, which Gillian wrote out next to the diagrams. There was explicit discussion of the links between both types of representation, and this was partly prompted by a comment from the teaching assistant, who called out, “Apparently some children can just see fours?”. This helped move the discussion along and enabled Gillian to demonstrate to the rest of the class how the rectangular diagram related to the abstract equation written on the board. In this section of the lesson there were high levels of pupil collaboration and discussion the whole way through, despite it being mostly whole class discussion and not peer to peer work. After approximately ten minutes of discussion in this section, the lesson was more than halfway through (about 38 minutes had passed). The teacher then asked the pupils to write about what they had been doing in their journals. In their journals, pupils seemed happy to use a variety of representations and whilst they were doing this the teacher was walking around having individual discussions with them about what they were doing. See the images below for some examples of pupil journals:





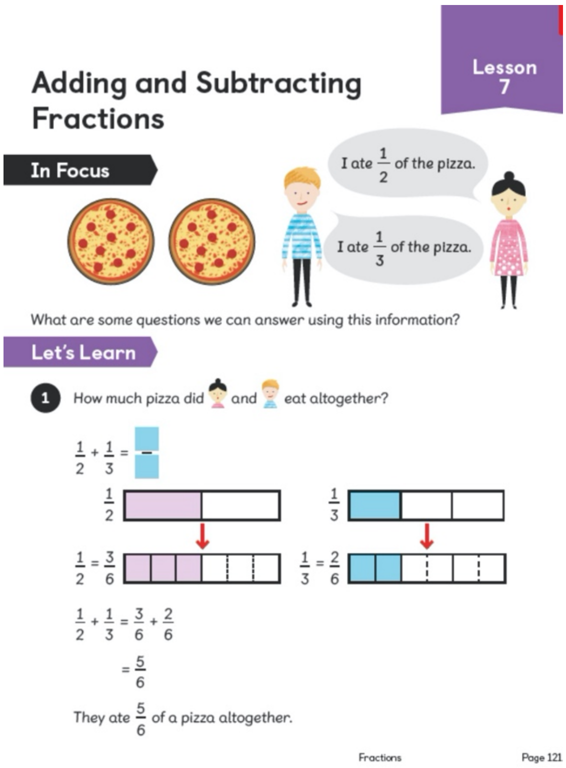
There was not much discussion at this point, although occasionally pupils spoke to each other about what they were doing. There was much less talk at this point though when compared to the rest of the lesson that had come before. During this time, Gillian moved around discussing what pupils were doing as they journal. Listening in to these discussions it seemed as if, for the most part, she did not confirm or deny pupil ideas, rather she was prompting them to think about what they had done with questions. Occasionally she reminded them of something discussed earlier, or in a previous lesson, which seemed to be a way of providing them with feedback. Throughout, Gillian’s stance as the teacher seemed to be one where the ideas of the pupils were valued on a similar level to her own ideas.

Once the lesson had been going on for approximately 48 minutes, Gillian asked the pupils to stop journaling. Although, despite this request, some pupils carried on with writing in their journals and Gillian seemed happy for this to happen, she did not stop them. She then showed some questions entitled ‘guided practice’ on the board (see textbook images above) and asked pupils to discuss the first one. As she was going through these questions, she allowed approximately two minutes of pupil discussion for each one and then they talked about them as a whole class with her asking questions like, “can someone jump into…’s brain to see what he is thinking?”. The main representations shown were bar models and abstract mathematical symbols. Whilst going through some of these Gillian appeared to encourage them to visualise by saying things like, “Can you just see it? Can you just see them?” By the time they were going through their third guided practice question they were only using the symbolic representation and seemed to be comfortable with this. This part of the lesson contained lots of discussion and the pupils were noisy – Gillian responds by saying, “love that you are heckling me!”. As an observer, it surprised me how much pupil talk there was, and that Gillian seemed happy with this. The lesson finished after the third guided practice question and the teacher told the pupils that they would need to complete workbook pages that go with it for ten minutes after their break.

The lesson concluded with Gillian briefly explaining what they had covered in the lesson and then telling them that they would practice simplifying fractions a little bit later that day. This means that Gillian was intending on providing time later in the day for pupils to complete some independent practice in their workbooks, which are another aspect of the textbook scheme she was using.

9.6.2 Lesson Observation 2

This second lesson occurred in late November 2019 and was the seventh lesson in a series of lessons about fractions (the same series as the first lesson observation). One of the first things I noticed was that Gillian was teaching lesson seven from the textbook, but over two weeks had passed since I observed her teacher lesson two. This meant that she had taken at least two school maths lessons on average to cover the content provided in each individual textbook lesson. I inferred from this that the pupils had been finding the content difficult and Gillian confirmed this in a later interview. The Maths – No Problem!™ (Ban Har et al., 2014) textbook pages that Gillian was using as the basis of this lesson are copied below.



Adding and Subtracting Fractions

Lesson 7

In Focus

I ate $\frac{1}{2}$ of the pizza.

I ate $\frac{1}{3}$ of the pizza.

What are some questions we can answer using this information?

Let's Learn

1 How much pizza did and eat altogether?

$\frac{1}{2} + \frac{1}{3} = \frac{\square}{\square}$

$\frac{1}{2}$ $\frac{1}{3}$

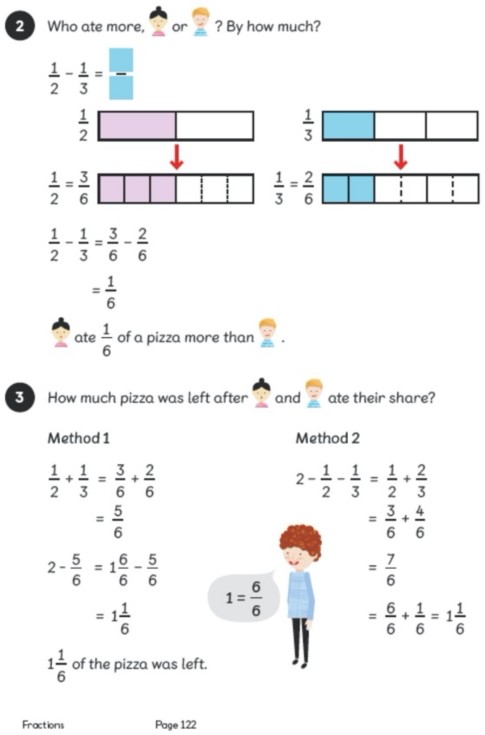
$\frac{1}{2} = \frac{3}{6}$ $\frac{1}{3} = \frac{2}{6}$

$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6}$

$= \frac{5}{6}$

They ate $\frac{5}{6}$ of a pizza altogether.

Fractions Page 121



2 Who ate more, or ? By how much?

$\frac{1}{2} - \frac{1}{3} = \frac{\square}{\square}$

$\frac{1}{2}$ $\frac{1}{3}$

$\frac{1}{2} = \frac{3}{6}$ $\frac{1}{3} = \frac{2}{6}$

$\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6}$

$= \frac{1}{6}$

ate $\frac{1}{6}$ of a pizza more than .

3 How much pizza was left after and ate their share?

Method 1

$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6}$

$= \frac{5}{6}$

$2 - \frac{5}{6} = 1\frac{6}{6} - \frac{5}{6}$

$= 1\frac{1}{6}$

$1\frac{1}{6}$ of the pizza was left.

Method 2

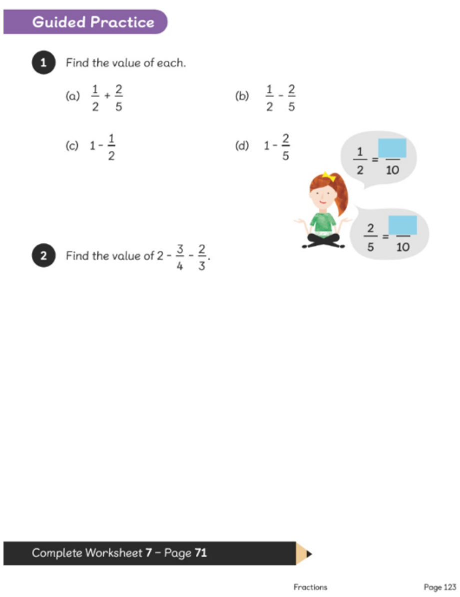
$2 - \frac{1}{2} - \frac{1}{3} = \frac{1}{2} + \frac{2}{3}$

$= \frac{3}{6} + \frac{4}{6}$

$= \frac{7}{6}$

$= \frac{6}{6} + \frac{1}{6} = 1\frac{1}{6}$

Fractions Page 122



Guided Practice

1 Find the value of each.

(a) $\frac{1}{2} + \frac{2}{5}$ (b) $\frac{1}{2} - \frac{2}{5}$

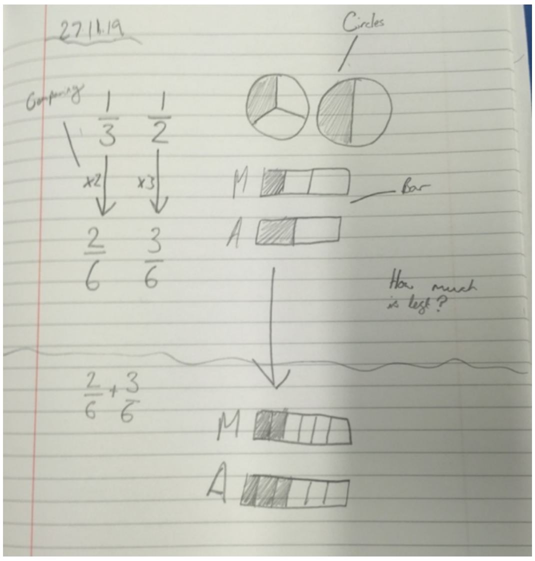
(c) $1 - \frac{1}{2}$ (d) $1 - \frac{2}{5}$

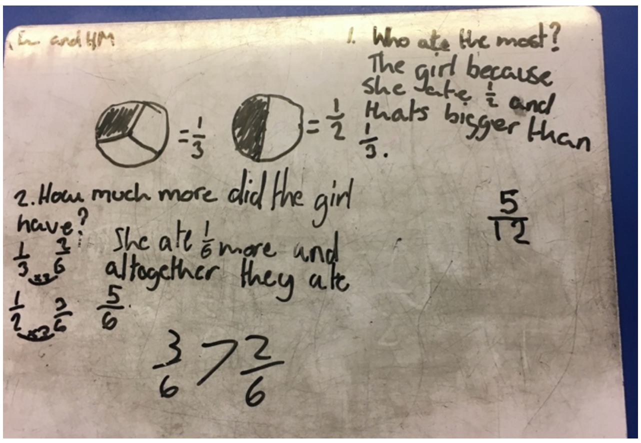
2 Find the value of $2 - \frac{3}{4} - \frac{2}{3}$.

Complete Worksheet 7 – Page 71

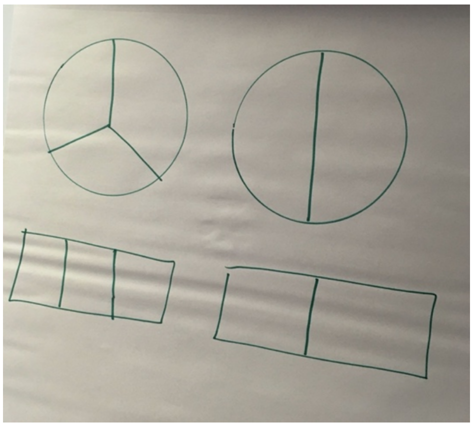
Fractions Page 123

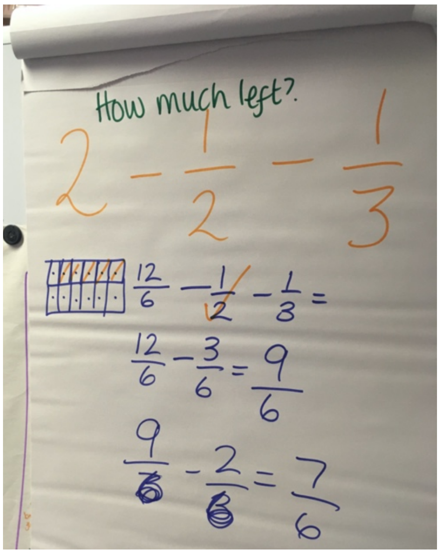
The lesson began with Gillian showing the image of the ‘In Focus’ task from the textbook on her whiteboard and asking pupils to represent the information in any way they could on their mini whiteboards. During this time, Gillian walked around asking pupils questions and there was a lot of pupil discussion. After a few minutes, she interrupted the pupils and drew attention to two specific representations she had seen pupils using – representing the pizzas as circles and representing them as rectangles (both split up into the given fractions in the problem). She then asks the question, “Why would some children decide to change the circular pizzas into rectangles?” As an observer I could see that she was trying to encourage them to move away from using circular representations because these would be less helpful in supporting their mathematical thinking. Gillian then asks the question, “So, who ate the most pizza?” and one pupil commented that “we already know that because we can just see it!” I could see that the visual representations were making the concept of fractions both visible and accessible to the pupils. She then prompted the pupils to use the information to come up with their own problems by drawing attention to the written question beneath the ‘In Focus’ task image. During this time there was a lot of pupil discussion, and they were writing their ideas down either on mini whiteboards or on paper jotters (see images below).



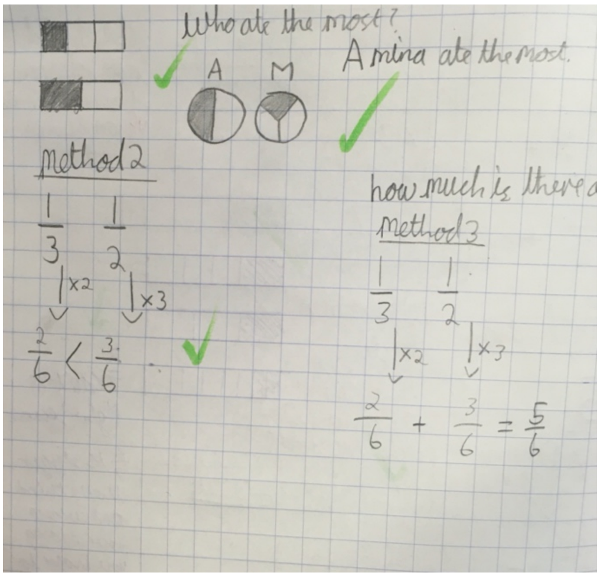


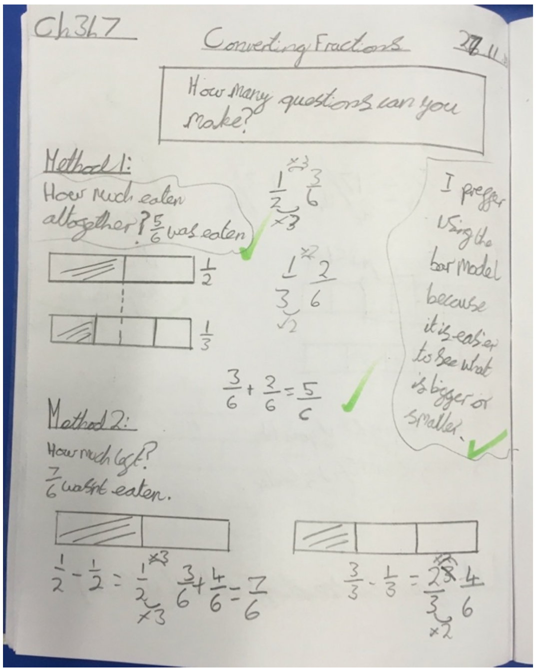
This first part of the lesson lasted for about 10 minutes, and I found it surprising that Gillian used this much time allowing the pupils to come up with any problem they wanted instead of steering their ideas towards addition and subtraction, which the rest of the textbook lesson focuses upon. After these first ten minutes, Gillian draws attention to a pupil who has come up with a problem that is about subtraction of the fractions. They spend about 20 minutes discussing this as the pupils appear to struggle with it. Many of them subtract the two fractions from one pizza initially, getting the answer ‘one sixth’ instead of subtracting from two whole pizzas. Gillian spends considerable time guiding whole class discussion about this, using different representations to try and help pupils understand the process (see images from her flipchart below). During this part of the lesson, the pupils become quieter and more focussed upon listening to the teacher, although there is still some pupil discussion. I felt that this was reflective of the difficulty that many pupils were having with the lesson content.

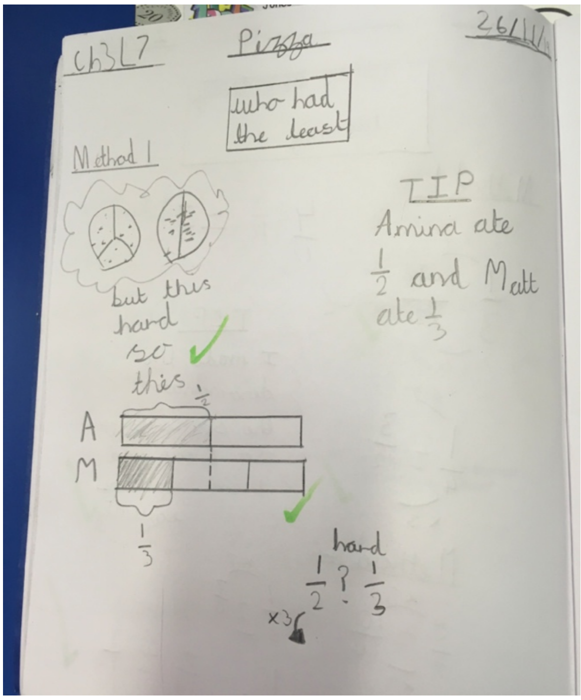




At this point, the lesson had been going for about 35 minutes and some pupils still seemed to be struggling to understand the process of subtracting two fractions from a whole number. Quite a few pupils call out and say things like, “This is really difficult!” I noted that, despite this difficulty, pupils still seemed motivated and happy to engage in the lesson. Gillian appeared to be aware of this difficulty and stated that, “I’m going to pause there because I think we need a bit of time to reflect.” She then asked them to get their journals out and show what they had been thinking about so far. As an observer I felt that Gillian was trying to get a better grasp of the pupils' understanding at this point in the lesson and was using the journals as a tool for assessment. As the pupils wrote and drew in their journals, Gillian walked around observing them and providing them with feedback about what they were doing. There was still quite a lot of pupil discussion as they were journaling (see images of journals below).







During this journaling time some pupils struggle in particular. Gillian prompted these children to go and speak to another child who seems to be more comfortable with the lesson content and reminds them ‘not to tell… just give some tips’. After just over ten minutes of journaling time, Gillian stopped the class and announced that she felt it was time for them to have a break. At this point the lesson finished.

9.7 Appendix 7: A sample of the vertical textbook analysis of fractions

Vertical Analysis – Fractions in the year 6 Maths – No Problem!™ (Ban Har et al., 2014) textbooks
The unit of analysis is each mathematical problem (including worked examples) in the textbook. So, an 'In Focus' task combined with its ensuing worked example counts as one task. Any other numbered task or worked example also counts as one problem.

Number of textbook 'lessons' in chapter – 16
Number of individual tasks in chapter – 58
Number of individual tasks in workbook – 43

Representation Combinations	*Textbook (58 tasks in total)*
Linked to problem context & maths, area model, standard maths symbols, words	4
Linked to problem context & maths, area model, standard maths symbols	5
Area model, standard maths symbols	13
Area model, standard maths symbols, words	1
Standard maths symbols only	27
Linked to problem context & maths, quantity representation, standard maths symbols	5
Linked to problem context & maths, quantity representation, area representation	1
Standard maths symbols and a written prompt to draw diagrams	1
Standard symbols and non-standard symbols	1
Quantity image alone	0
Linear	0
Decorative only	0
Linked to problem context but not mathematics	0
Prompted use of manipulatives	0
Task presented in written form	55

Construct	Out of 16 lessons
Part-part whole	7
Operator (Transforms lines, figures or numbers)	6
Ratio (comparison)	0
Measure	3
Quotient (Division of two whole numbers)	0

Part-part whole and Operator combined	0

Workbook representations
In the fractions workbook pages, the vast majority of tasks use standard mathematical symbols only. Only 7 out of 43 tasks had another type of representation and these were all area model alongside standard maths symbols. It is worth noting that the tasks with an area model are at the start of the chapter (lessons 1, 2 and 3) and then re-appear when the lessons move on to focusing on an operator construct mid-way through the chapter.

Fractions later in the Y6 Maths – No Problem!™ (Ban Har et al., 2014) textbook

Fractions appear at several points in other chapters after the fractions chapter within the Y6 textbook.

P. 154 – within the first lesson on decimals, fractions are used to demonstrate a decimal as a smaller part of a whole (the whole being 1). E.g. 1/100 = 0.01. The link is also made to these represented as words ('one tenth, one hundredth etc.) and this continues in the chapter suggesting a strong focus on language being used to support the development of understanding and making connections (e.g. 'one tenth' is 1/10 and also 0.1).

P.162 In the decimals chapter there is a lesson on dividing whole numbers to make decimals and there is a link made to fractions here (fractions as quotients). This leads into a sequence of 3 lessons where the focus is on writing fractions as decimals and the underlying construct is fractions as a quotient.

P.198 In the chapter on measurements, fractions are frequently used alongside decimals to demonstrate how to convert units of measurement. (e.g. 25m = 25/1000km = 0.025km)

p. 206 A lesson on time – fractions are used to represent an amount of time in hours (e.g. 2h 12min = 2 1/5 hours) This presents quite a challenging task for pupils as they will need to consider fractions as an operator (what fraction is 1/5 of 60 minutes) but also do this in reverse (What fraction of 60mins is 12 minutes?) This is verging on use of fractions as ratio which comes later in the textbook (see below).

Word problem chapter – there is one lesson focusing on word problems with fractions as this contains a significant amount of rectangular area models alongside words and standard maths symbols. It is worth noting that in previous lessons, the exact same rectangular model (bar model) is used with multiplication and division word problems – is this designed to help pupils make more connections to different areas of maths?

p. 6 (6B) in the percentages chapter, the fraction 25/100 is used to represent 25%. This is then connected to ¼ to help draw further connections. Rectangular area models similar to in the fractions chapter are also used for percentage.

p.12 There is a lesson on using percentage to compare amounts and fractions are used alongside decimals in this lesson as well. This is an example of fractions used as a ratio construct.

p.16 – shows measure (money) connected to decimals, fractions and percentage.

Ratio chapter – there are two lessons in this chapter that use fractions (standard symbolic representation) alongside ratio. These lessons also use a rectangular area model and are using fractions as a ration construct.

p. 61 – in the algebra chapter, there is a lesson on algebraic expressions using the idea of a 'number crunching machine'. Here fractions are brought in as operators and also the idea of a fraction with an unknown numerator is introduced ($x \div 3 = x/3$). This idea appears again in a subsequent lesson briefly (p. 73) and also in the mind workout at the end of the algebra chapter.

Area & Perimeter chapter – Fractions are used as operators when looking at the different ways to calculate the area of triangles and parallelograms.

p. 110 – Fractions are used as operators in calculating the volume of cuboids.

P.136 – Fractions used as a measure in some numbers when calculating the length of a circle's diameter.

p. 202 – Fractions are used in the context of coordinates – this is a more complex example of fractions being represented in a linear fashion.

p. 221 – Fractions used as a measure within a lesson about averages. A pictogram representation is used using footballs (like a circular area model) where 4 ¼ is shown as 4 whole balls and one quarter ball.

P. 251 – Fraction used in a linear way as part of reading a line graph.